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SURFACE BOUNDARY CONDITIONS TRIGGER FLUTTER INSTABILITY IN NON-ASSOCIATIVE ELASTIC–PLASTIC SOLIDS

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Abstract—It is shown that *surface flutter* instability may be triggered by the *simultaneous* influence of non-associativity and boundary conditions even if, taken independently, neither the elastic–plastic constitutive law (satisfying deviatoric associativity) nor the boundary conditions (no applied traction rates) would lead to flutter. More specifically, it is shown that, for orthotropic elastic–plastic constitutive tensors with an orthotropy axis tangent to the rate-traction-free boundary, the onset of surface flutter instability coincides with the *incipience* of plasticity for *any* non-associative flow rule, whenever the normal to the boundary does not coincide with one of the other two orthotropy axes. In the case of deviatoric associativity and coaxiality between the directions of orthotropy and the normal and tangent to the rate-traction-free boundary, surface flutter instability may also occur, but only for unusual values of material parameters.

Additional results and discussion are also presented for the onset of *stationary* surface waves in associative and non-associative elastic–plastic bodies. In contrast to the onset of surface flutter instabilities, the condition for the onset of stationary surface waves involves material properties only, that is, it does not discriminate among different orientations of the rate-traction-free boundary with respect to the material orthotropy directions.

1. INTRODUCTION

The well-posedness of the initial-boundary-value problem of the mechanics of solids requires the dynamical equations of motion to be pointwise hyperbolic. Equivalently, the three acceleration wave-speeds must be real and strictly positive [see, for example Truesdell and Noll (1965), Section 71].

The squares of the latter are the eigenvalues of the acoustic tensor, which inherits the major symmetry property from the tensor of constitutive moduli. Consequently, the squares of the wave-speeds are real for elastic–plastic solids with associative flow rules, and loss of hyperbolicity occurs when the smaller wave-speed vanishes (Hill, 1962), a phenomenon referred to as a *stationary discontinuity*. The quasi-static interpretation of a stationary discontinuity as a *strain localization* has been exposed by Mandel (1962) and Rice (1976) for materials with both associative and non-associative flow rules. Both Mandel and Rice pointed out the destabilizing influence of non-associative flow-rules; Asaro and Rice (1977) investigated the destabilizing effects of various constitutive features that lead to loss of the major symmetry of the tensor of constitutive moduli. The above works have been at the root of a number of papers, whose essential aim was to improve upon the somewhat resistant behaviour of classic elastic–plastic solids that yield too late an onset of strain localization as compared with experimental data.

For constitutive equations that lack the major symmetry, the acoustic tensor is not symmetric, and loss of hyperbolicity may occur through a mode other than a stationary discontinuity. This mode is termed, after Rice's terminology, a *flutter instability*: it corresponds to the square of a wave-speed becoming complex. It has been the subject of a few investigations so far. Loret *et al.* (1990) show that it is excluded for a wide range of elastic–plastic solids, namely, those whose flow rules are associative with respect to the deviatoric components of the plastic strain rate tensor only, a constitutive feature referred to as *deviatoric associativity*. Ottosen and Runesson (1991) improve slightly upon the former result by showing that, if the unit normals to the plastic potential and to the yield surface are coaxial, with their eigenvalues arranged in the same order, then once again flutter instability is excluded. A route to develop constitutive equations that may lead to the onset of flutter instability has been explored by An and Schaeffer (1992): in the analysis of two-dimensional models, they show that (even for associative flow rules) strict hyperbolicity does not hold; thus it might be possible to devise constitutive perturbations involving loss of the major symmetry of the constitutive moduli and leading to the onset of flutter instability. This idea is reconsidered in the usual three-dimensional elastic–plastic context by Loret (1992). He considers constitutive perturbations of solids with deviatoric associativity and obtains the following results, which, interestingly enough, bear much qualitative resemblance to the work reported in this paper: any infinitesimal constitutive perturbations from deviatoric associativity, making the normals to the plastic potential $\mathbf{P}$ and to the yield surface $\mathbf{Q}$ non-coaxial, lead to the onset of flutter instability for a properly chosen plastic modulus; in contrast, a finite amount of deviation from deviatoric associativity is required to reach the onset of flutter when the normals to the plastic potential and the yield surface are kept coaxial. Notice that this last result is in agreement with the work of Ottosen and Runesson (1991): that finite amount of deviation from deviatoric associativity is the amount necessary to change the order of the eigenvalues of $\mathbf{P}$ and $\mathbf{Q}$.

Notice that the above results are local, that is, they may be viewed as concerning a point of an infinite solid. Another route to the onset of flutter instability may therefore be the incorporation of the effects of boundary conditions on a finite or a semi-infinite body. This is the investigation that we envisage in this paper. One motivation for this study stems from previous analyses of the strongly destabilizing effects of Coulomb's friction law. Martins *et al.* (1992) consider the motion of a rigid half-plane over a linear isotropic half-plane with frictional contact. For a sufficiently large friction coefficient and Poisson's ratio, growing oscillations develop at points just below the contact line, a situation similar to that of Rayleigh waves with a complex wave-speed, that is, a *surface flutter instability*. Similar results are obtained when the deformable half-plane is viscoelastic (Martins *et al.*, 1994) or when finite deformations are accounted for through neo-Hookean elastic behaviour (Martins and Faria, 1991). These observations may contribute to an explanation of the origins of the Schallamach waves, which act as "waves of detachment" that cross the contact area from front to rear with rapidly alternating states of contact and loss of contact (Schallamach, 1972).

Clearly, flutter instability requires the governing equations to possess some kind of non-self-adjointness, although, as pointed out above, non-self-adjointness is not a sufficient condition for the onset of flutter instability in elastic–plastic solids. The lack of symmetry of the tensor of constitutive moduli implies the dynamic equations of motion to be non-self-adjoint, whereas, for frictional contact problems, non-self-adjointness is induced by the boundary conditions. Here we adopt a somewhat intermediate position by considering an elastic–plastic solid with deviatoric associativity filling a half-plane with a rate-traction-free boundary. So, *taken independently*, neither the constitutive equations nor the boundary conditions lead to the onset of flutter instability. However, we shall derive conditions under which their *simultaneous* influence will indeed lead to the onset of flutter instability.

We are not aware of any other attempt to exhibit flutter instability due to boundary effects. On the other hand, this analysis can be viewed as an instance where an initial-boundary-value problem becomes ill-posed while the rate-field equations are still hyperbolic. In that sense, contact can be made with analyses of the existence of stationary waves in bounded or semi-infinite solids. For example, Benallal *et al.* (1989) note that failure of the complementing condition is equivalent to the existence of stationary surface waves. Conditions for the existence of these waves are analysed by Dowaikh and Ogden (1990) for particular pre-stressed incompressible elastic solids and by Needleman and Ortiz (1991) for elastic–plastic solids. The existence of stationary *interfacial* waves between two perfectly bonded half-spaces is analysed by Dowaikh and Ogden (1991), and some connection with the conditions for existence of stationary surface waves is established; this problem is also considered by Needleman and Ortiz (1991). Suo *et al.* (1992) show that, when the interface between two elastic–plastic half-spaces is compliant and involves a length-scale, the onset of stationary interfacial waves depends on the wavelength and there exists in some circumstances a minimum wavelength for the instability mode.

This paper is organised as follows. In Section 2, we present the field and boundary conditions, and we obtain a variational formulation for the problem of finding dynamic surface solutions in an elastic–plastic half-space where plastic loading holds pointwise while the applied traction rates at the boundary are null. The analysis is restricted to orthotropic tensors of constitutive moduli. The resulting variational formulation trivially leads to the expected result that surface flutter instability is ruled out for associative flow rules.

The analysis delineates two cases: the coaxial case, in which one orthotropy direction is normal and the others are tangent to the free surface, and the non-coaxial case, in which only a tangent to the free surface coincides with one of the orthotropy directions. Section 3 is devoted to the analysis of the coaxial situation. We first describe the constitutive equations corresponding to an elastic–plastic solid with deviatoric associativity (Section 3.1), and then we obtain the field and boundary conditions for motions that are obtained by the superposition of surface modes (Section 3.2). To be relevant, flutter instability must occur prior to body or surface stationary waves. The onsets of these are given an explicit form in Sections 3.3 and 3.4. However, it is not possible to give an explicit expression for the plastic modulus at the onset of flutter instability. Nevertheless, we are able to show that this onset may be characterized by a pair of inequalities involving only material parameters (Section 3.5). To establish this result we use some properties of the solutions of a biquadratic equation with complex coefficients established in Appendix A. The conclusion drawn from this coaxial case is somewhat disappointing, since, quantitatively, it shows that flutter is excluded for the usual constitutive parameters (Section 3.6).

The analysis of the non-coaxial case presented in Section 4 leads to our main result, which is in strong contrast with the coaxial case: we show that, in the non-coaxial case, *the onset of flutter instability coincides with the incipience of plasticity*, that is, the critical plastic modulus at the onset of flutter instability is infinite. This result holds for any type of non-associative flow rules, thus including deviatoric associativity. We therefore have an instance in which a semi-infinite body may undergo flutter instability whereas the infinite body filled with the same material does not (recall the result on solids with deviatoric associativity reported by Loret *et al.* (1990)).

Specifically, we assume that, at the current stress, the yield surface and the plastic potential are smooth; moreover, we neglect co-rotational terms in the stress derivative involved in the rate-constitutive equations. This restriction is not thought to be of practical importance for many applications in civil engineering and geomechanical analyses. However, the results presented here are not expected to apply in the case in which the objective stress-rate introduces terms in the constitutive moduli that break the major symmetry property as, for example, the Jaumann rate does.

It should also be clear that the present work is devoted to defining the onset of flutter instability. The question of whether the phenomenon will occur or not during specific loading processes is not addressed here. In addition, the exclusive consideration of the plastic loading regime is consistent with the linearized stability analysis performed here, but it precludes an assessment of the full consequences of the instabilities detected, particularly when they are of the flutter type. Indeed, in this case, the growing oscillatory nature of the solutions may be expected to lead in relatively short times to situations of local plastic unloading, which, of course, are quite apart from the linearized loading regime. Similarly, the validity of the analysis of Martins and his co-workers ceases, strictly speaking, if contact is lost or if (unloading) stick occurs. From a linearized dynamic stability analysis in the (loading) slip regime, it is not possible to conclude if the dynamic instability gives rise to an oscillatory phenomenon much like stick/slip or if it has negligible consequences in the overall behaviour of the system.

The reader may wonder why we consider independently the so-called coaxial and non-coaxial cases, although the latter case embodies the former one. We proceed in this manner

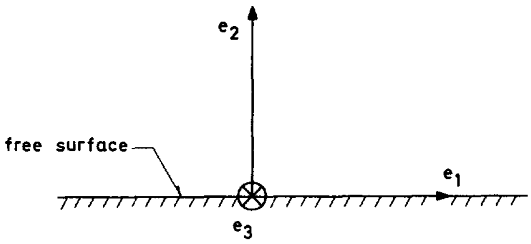


Fig. 1. The lower half-plane is filled with an elastic–plastic solid, and the boundary $x_2 = 0$ is rate-traction-free at any time.

for clarity of presentation. Actually, the results obtained in the coaxial and non-coaxial cases are quite different, and so are the analytical developments leading to them.

2. PROBLEM STATEMENT

2.1. *Governing equations*

Let us consider a semi-infinite and homogeneous elastic–plastic body with a rate-traction-free boundary. Let $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ (see Fig. 1) be a fixed orthonormal reference frame with the origin on the boundary of the body. Here $\mathbf{e}_1$ is tangent to the free surface, $\mathbf{e}_2$ is the outward normal, and $\mathbf{e}_3$ is the out-of-plane direction. We denote by $\mathbf{x} = (x_1, x_2, x_3)$ the position vector of the particles of the body.

Since the analysis assumes that the material undergoes small perturbations relative to a given reference state, all time derivatives coincide with the material time derivative denoted by a superposed dot.

The material is isotropic in its elastic properties, and we shall restrict the analysis to orthotropic elastic–plastic solids. The requirement that the elastic strain-energy function is positive-definite places a restriction on the tensor of elastic moduli $\mathbf{E}^e$ that allows one to normalize the Lamé modulus Λ, the stress tensor $\boldsymbol{\sigma}$, and the mass density ρ by dividing them by the elastic shear modulus μ. Thus formally $\mu = 1$, and the inequality

$$\Lambda + \tfrac{2}{3} > 0 \tag{1}$$

holds, which is known to imply the existence of one elastic shear wave with speed c_S^e (of multiplicity two) and one elastic longitudinal wave with speed c_L^e such that

$$c_S^e = \left[\frac{1}{\rho}\right]^{1/2} < c_L^e = \left[\frac{\Lambda+2}{\rho}\right]^{1/2}. \tag{2}$$

The tensor of elastic–plastic moduli $\mathbf{E}$ has the form

$$\mathbf{E} = \mathbf{E}^e - \frac{1}{H}(\mathbf{E}^e : \mathbf{P}) \otimes (\mathbf{Q} : \mathbf{E}^e), \tag{3}$$

where $\mathbf{P}$ and $\mathbf{Q}$ are the unit outward normals to the smooth plastic potential g and to the smooth yield surface f, respectively. Here, the symbol $\otimes$ denotes the dyadic product, and a double dot product represents the tensor inner product between adjacent dyads (so that, in Cartesian components, adjacent indices are summed pairwise). The modulus H is restricted to strictly positive values in order to exclude locking materials and is given by

$$H = h + h_e, \tag{4}$$

where h is the plastic modulus and

$$h_c = \mathbf{P} : \mathbf{E}^e : \mathbf{Q}. \tag{5}$$

For a given velocity field $\mathbf{v}$ with gradient $\nabla\mathbf{v}$, the material response is plastic if, simultaneously, f is zero and $\mathbf{Q} : \mathbf{E}^e : \nabla\mathbf{v}$ is positive; the resulting stress rate $\dot{\boldsymbol{\sigma}}$ is then given by

$$\dot{\boldsymbol{\sigma}} = \mathbf{E} : \nabla\mathbf{v}. \tag{6}$$

Otherwise, the material response is elastic and

$$\dot{\boldsymbol{\sigma}} = \mathbf{E}^e : \nabla\mathbf{v}. \tag{7}$$

In this analysis, we shall assume that plastic loading (eqn 6) holds pointwise.

It is of importance to emphasize that only for associative plastic flow rules does the tensor of elastic–plastic moduli enjoy the major symmetry $E_{ijkm} = E_{kmij}$; on the other hand, it always has the minor symmetries $E_{ijkm} = E_{jikm}$ and $E_{ijkm} = E_{ijmk}$ $(i, j, k, m = 1, 2, 3)$.

In this paper, we shall be concerned with the propagation of small plane disturbances orthogonal to $\mathbf{e}_3$, which is assumed to be an *axis of orthotropy* of the elastic–plastic material response. The problem that we wish to solve is thus to find a plane velocity field $\mathbf{v}(x_1, x_2, t) = v_1(x_1, x_2, t)\,\mathbf{e}_1 + v_2(x_1, x_2, t)\,\mathbf{e}_2$ $(v_3 = 0)$ in the half-plane $x_2 \leqslant 0$ such that, for all times t, the rate form of the (two) equations of linear momentum balance

$$\dot{\sigma}_{ij,i} = \rho\ddot{v}_j \tag{8}$$

holds for $x_2 < 0$ and the rate-traction-free boundary conditions

$$\dot{\sigma}_{21}(x_1, 0, t) = 0, \quad \dot{\sigma}_{22}(x_1, 0, t) = 0 \tag{9}$$

hold on $x_2 = 0$. In eqn (8) and from now on, unless stated otherwise, latin indices i, j, k, m run from 1 to 2, the summation convention over repeated indices is in force, and a lower comma denotes a partial spatial derivative.

Using the constitutive relation (6), we obtain the system of two partial differential equations

$$E_{ijkm} v_{k,mi} = \rho\ddot{v}_j \tag{10}$$

for $x_2 < 0$, together with the two boundary conditions at $x_2 = 0$

$$E_{2jk1} v_{k,1}(x_1, 0, t) + E_{2jk2} v_{k,2}(x_1, 0, t) = 0. \tag{11}$$

2.2. *Surface solutions*

In the present analysis, we seek surface solutions $\mathbf{v}$ to eqns (10) and (11) in the form

$$v_k = V_k(n_1 x_2)\exp(in_1(x_1 - ct)). \tag{12}$$

Here n_1 is a positive real number that represents the angular frequency of the solutions along the $\mathbf{e}_1$ axis; $V_k(n_1\,x_2)$ are sufficiently smooth functions defined on $]-\infty, 0]$ with an appropriate decay as $x_2 \to -\infty$, namely

$$\lim_{x_2 \to -\infty} V_k(n_1 x_2) = 0; \tag{13}$$

c is a complex number such that Re (c) represents the speed at which the solution in eqn (12) propagates along the $\mathbf{e}_1$ axis and n_1 Im (c) represents the rate of exponential growth or decay of that solution in time. Note that the well-known (surface) *Rayleigh waves* in an elastic half-space are of the form of eqns (12) and (13) with c^2 real and positive. Here, we shall be particularly interested in detecting the situations in which, by influence of the non-associative elastic–plastic behaviour, c^2 ceases to be a positive real number.

Throughout the paper, we adopt the usual notation for complex numbers, namely, i is such that $i^2 = -1$, Re (z) and Im (z) denote the real and imaginary parts of a generic complex number $z = \text{Re}\,(z) + i\,\text{Im}\,(z)$, and $\bar{z} = \text{Re}\,(z) - i\,\text{Im}\,(z)$ is the complex conjugate of z.

Inserting eqn (12) in eqn (10), we obtain a system of linear and homogeneous second-order ordinary differential equations

$$\mathbf{A}\mathbf{V}'' + \mathbf{B}\mathbf{V}' + (\mathbf{C} - X^2\mathbf{I})\mathbf{V} = \mathbf{0}, \tag{14}$$

where **A**, **B**, and **C** are 2×2 matrices defined componentwise by

$$A_{jk} = -E_{2jk2}, \tag{15}$$

$$B_{jk} = -i(E_{1jk2} + E_{2jk1}), \tag{16}$$

$$C_{jk} = E_{1jk1} \tag{17}$$

and **V** is the vector of length 2 with components $V_k(n_1 x_2)$. The symbol **I** is used to denote the unit tensor of appropriate order or its matrix representation, a superposed prime implies a derivative with respect to $n_1 x_2$ and

$$X^2 = (c/c_S^e)^2. \tag{18}$$

Note that eqn (14) is the dynamic counterpart of equation (IV.2.13) of Benallal *et al.* (1993). On inserting eqn (12) in eqn (11), the boundary conditions become

$$iE_{2jk1} V_k(0) + E_{2jk2} V_k'(0) = 0. \tag{19}$$

2.3. *Variational formulation*

The above eigenproblem can be given a compact and precise variational formulation. In this respect, note that, in general, the eigenfunctions V_k may be complex functions: an oscillatory behaviour along the x_2 axis is admissible provided that the decay condition, eqn (13), is satisfied. On the other hand, the non-symmetry of the elastic–plastic tensor moduli may lead to complex eigenvalues X^2. Hence, instead of introducing the usual bilinear forms on spaces of real functions, we shall have to work with sesquilinear forms in spaces of complex functions.

Formally multiplying eqn (14) by the complex conjugates of arbitrary sufficiently smooth functions $W_k(n_1 x_2)$, $k = 1, 2$, with compact support in $]-\infty, 0]$, integrating the result between $-\infty$ and 0, and performing an integration by parts of the terms involving E_{2jk1} and E_{2jk2}, we obtain:

$$\int_{-\infty}^{0} [E_{2jk2} V_k' \bar{W}_j' + i(E_{2jk1} V_k \bar{W}_j' - E_{1jk2} V_k' \bar{W}_j) + E_{1jk1} V_k \bar{W}_j]\, dx_2$$
$$+ (iE_{2jk1} V_k(0) + E_{2jk2} V_k'(0))\bar{W}_j(0) = X^2 \int_{-\infty}^{0} \bar{W}_k V_k\, dx_2 \tag{20}$$

for all such vector functions **W**. To arrive at eqn (20), use has already been made of the fact that each test function **W** is null for $|x_2|$ sufficiently large. Taking now the boundary conditions (eqn (19)) into account, we obtain the variational equation:

$$a(\mathbf{V}, \mathbf{W}) = X^2(\mathbf{V}, \mathbf{W}), \tag{21}$$

where

$$(\mathbf{V}, \mathbf{W}) = \overline{(\mathbf{W}, \mathbf{V})} = \int_{-\infty}^{0} \bar{W}_k V_k \, \mathrm{d}x_2 \tag{22}$$

and

$$\begin{aligned} a(\mathbf{V}, \mathbf{W}) &= \int_{-\infty}^{0} [E_{2jk2} V'_k \bar{W}'_j + \mathrm{i}(E_{2jk1} V_k \bar{W}'_j - E_{1jk2} V'_k \bar{W}_j) + E_{1jk1} V_k \bar{W}_j] \, \mathrm{d}x_2 \\ &= \int_{-\infty}^{0} (\overline{\mathrm{i} W_i I_{1j} + W'_i I_{2j}}) E_{ijkm} (\mathrm{i} V_k I_{1m} + V'_k I_{2m}) \, \mathrm{d}x_2 . \end{aligned} \tag{23}$$

The derivation above motivates the adoption of the following appropriate functional setting and statement for the eigenproblem at hand (I denotes the interval $]-\infty, 0[$):

Problem 2.1. Find the squares of the speed of propagation-like scalars $X \in \mathbb{C}$ and the functions $\mathbf{V} \in V = (H_1(I))^2$, $\mathbf{V} \neq \mathbf{0}$, such that eqn (21) holds for all $\mathbf{W} \in V$.

Note that the Hilbert space V is a space of complex functions on I, the symmetric sesquilinear form $(\mathbf{V}, \mathbf{W})$ defined by eqn (22) is the inner product on $(L^2(I))^2$, and $a(\mathbf{V}, \mathbf{W})$ is a continuous sesquilinear form on V. It is then possible to show, by using a generalized Green's formula [cf. Showalter (1977), p. 58], that any $\mathbf{V} \in V$ is a solution to Problem 2.1 if and only if it satisfies (in $(L^2(I))^2$) the system of second-order ordinary differential equations, eqn (14), and also satisfies the boundary conditions, eqn (19). In addition, any $\mathbf{V} \in (H_1(I))^2$ satisfies the decay condition of eqn (13) [cf. Brezis (1983), Corollaire VIII.8].

An important result follows easily from this variational statement. In the case of an associative plastic flow rule, the tensor of elastic–plastic moduli enjoys the major symmetry $E_{ijkm} = E_{kmij}$ and the sesquilinear form, eqn (23), is symmetric, i.e.

$$a(\mathbf{V}, \mathbf{W}) = \overline{a(\mathbf{W}, \mathbf{V})} \quad \forall \, \mathbf{V}, \mathbf{W} \in V. \tag{24}$$

Consequently, we have proved:

Proposition 2.2.: For associative plastic flow rules, the scalars X^2 that solve Problem 2.1 are necessarily real.

3. ELASTIC–PLASTIC SOLIDS COAXIAL WITH THE FREE BOUNDARY

In this section, the material response is assumed to satisfy deviatoric associativity and to be orthotropic, with the axes $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ as orthotropy axes. We derive the conditions that lead to the onset of flutter instability and we show, quantitatively, that flutter is excluded for the usual constitutive parameters.

We proceed as follows. We first detail the constitutive equations and the field and boundary conditions. We then derive the conditions that ensure that the field equations remain hyperbolic and that stationary surface waves are excluded: the underlying reason for this approach is that we are interested in surface flutter instabilities that occur before other surface or body instabilities.

3.1 *Constitutive equations*

When expressed in the orthotropy axes, the constitutive equations (6) take the form:

$$\left.\begin{aligned}\dot{\sigma}_{11} &= L_{11}\dot{\varepsilon}_{11}+L_{12}\dot{\varepsilon}_{22}\\ \dot{\sigma}_{22} &= L_{21}\dot{\varepsilon}_{11}+L_{22}\dot{\varepsilon}_{22}\\ \dot{\sigma}_{12} &= 2\dot{\varepsilon}_{12}\end{aligned}\right\}, \tag{25}$$

where $\dot{\boldsymbol{\varepsilon}}$ is the strain-rate tensor, given by:

$$\dot{\boldsymbol{\varepsilon}} = (\nabla\mathbf{v}+\nabla\mathbf{v}^{\mathrm{T}})/2. \tag{26}$$

For an elastic–plastic solid obeying deviatoric associativity, identification of the four scalar coefficients L_{11}, L_{22}, L_{12}, and L_{21} requires specification of three constitutive plastic functions (plastic modulus h, friction angle ψ, and dilatancy angle χ), a triaxiality angle (see eqn (32) below) and the elastic properties:

$$\left.\begin{aligned}L_{ii} &= E_{iiii} = \Lambda+2-r\,(N_{\chi}+\hat{S}_{ii})(N_{\psi}+\hat{S}_{ii})\,, && \text{(no sum on } i)\\ L_{ij} &= E_{iijj} = \Lambda-r\,(N_{\chi}+\hat{S}_{ii})(N_{\psi}+\hat{S}_{jj})\,, \quad i\neq j && \text{(no sum on } i\text{, no sum on } j)\end{aligned}\right\}, \tag{27}$$

with

$$N_{\chi} = \frac{\sqrt{3}}{2}(\Lambda+\tfrac{2}{3})\tan\chi\,, \quad N_{\psi} = \frac{\sqrt{3}}{2}(\Lambda+\tfrac{2}{3})\tan\psi, \tag{28}$$

$$r = \frac{4}{H}\cos\psi\cos\chi, \tag{29}$$

where H is defined by eqn (4) and h_{e} by eqn (5), namely:

$$h_{\mathrm{e}} = \mathbf{P}:\mathbf{E}^{\mathrm{e}}:\mathbf{Q} = 2\cos\psi\cos\chi+3(\Lambda+\tfrac{2}{3})\sin\psi\sin\chi. \tag{30}$$

To arrive at eqns (28)–(30), one has assumed the deviatoric parts of the unit normals to the plastic potential $\mathbf{P}$ and the yield surface $\mathbf{Q}$ to be the same, i.e.

$$\left.\begin{aligned}\mathbf{P} &= \cos\chi\,\hat{\mathbf{S}}+\sin\chi\frac{\mathbf{I}}{\sqrt{3}}\\ \mathbf{Q} &= \cos\psi\,\hat{\mathbf{S}}+\sin\psi\frac{\mathbf{I}}{\sqrt{3}}\end{aligned}\right\}, \tag{31}$$

where $\hat{\mathbf{S}}$ is a unit deviatoric tensor that may embody any kind of anisotropy. In the orthotropy axes, a single scalar $\theta\in[0,\pi/3]$ modulo $2\pi/3$ defines completely the unit deviator $\hat{\mathbf{S}}$:

$$\hat{S}_{ii} = \sqrt{\tfrac{2}{3}}\cos\left[\theta-\tfrac{2}{3}(i-1)\pi\right], \quad i = 1,2,3\ (\text{no sum on } i). \tag{32}$$

Notice that the assumption that the material response is orthotropic in the axes $(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$ is equivalent to $\hat{\mathbf{S}}$ being principal in these axes.

3.2. *Field equations and boundary conditions*

Solutions to the system of linear and homogeneous second-order ordinary differential equations (eqn (14)) are sought in the form of linear combinations of functions

$$V_k(n_1x_2) = U_k \exp(\mathrm{i}Nn_1x_2), \tag{33}$$

where U_k is an undetermined constant amplitude and $N = n_2/n_1$ is a complex number that satisfies

$$\mathrm{Im}\,(N) < 0. \tag{34}$$

Thus we allow the solution to oscillate along the $\mathbf{e}_2$-axis, but, in accordance with eqn (13), we require its amplitude to decay exponentially along this axis.

Satisfaction of the rate field equations (14) for a non-trivial mode $\mathbf{V}$ requires the following algebraic system in U_1 and U_2 to be indeterminate:

$$\left.\begin{aligned}(L_{11}+N^2-X^2)U_1+(L_{12}+1)N\quad U_2 &= 0\\ (1+L_{21})N\quad U_1+(L_{22}N^2+1-X^2)\quad U_2 &= 0\end{aligned}\right\}. \tag{35}$$

Thus the complex scalar N solves the biquadratic equation with complex coefficients

$$(N^2)^2-SN^2+P = 0, \tag{36}$$

where, assuming $L_{22} \neq 0$ (see Section 3.3 for justification),

$$\left.\begin{aligned}S &= \frac{1}{L_{22}}[-\Delta+(L_{12}+L_{21})+(1+L_{22})X^2]\\ P &= \frac{1}{L_{22}}(1-X^2)(L_{11}-X^2)\end{aligned}\right\}. \tag{37}$$

The scalar Δ defined below will be shown to play a cardinal role in the surface instability analysis:

$$\begin{aligned}\Delta &= L_{11}L_{22}-L_{12}L_{21}\\ &= 4(\Lambda+1)-4(\cos\psi\cos\chi)H^{-1}\{4N_\chi N_\psi+2(N_\chi+N_\psi)(\hat{S}_{11}+\hat{S}_{22})\\ &\quad+\Lambda(\hat{S}_{11}-\hat{S}_{22})^2+2(\hat{S}_{11}^2+\hat{S}_{22}^2)\}.\end{aligned} \tag{38}$$

Note that the complex speed of propagation-like scalar X is still undefined. To close the problem, we require the boundary conditions, eqn (19), at the surface $x_2 = 0$ to be satisfied. Since eqn (36) for N is biquadratic, it has at most two solutions N satisfying the decay condition, eqn (34); hence we can construct $\mathbf{V}\,(n_1x_2)$ by superposition of two modes of the form of eqn (33):

$$\mathbf{V}(n_1x_2) = \mathbf{U}^{(1)}\exp(\mathrm{i}N^{(1)}n_1x_2)+\mathbf{U}^{(2)}\exp(\mathrm{i}N^{(2)}n_1x_2). \tag{39}$$

Here, each $N^{(\alpha)} = n_2^{(\alpha)}/n_1$, $\alpha = 1, 2$, is precisely one of the two solutions of eqn (36) satisfying the decay condition, eqn (34); the corresponding $\mathbf{U}^{(\alpha)}$, $\alpha = 1$, 2, satisfies eqn (35) with $N = N^{(\alpha)}$.

Inserting this kinematic in the boundary conditions, eqn (19), one obtains a second linear system for the scalars $U_1^{(\alpha)}$ and $U_2^{(\alpha)}$:

$$\left.\begin{aligned} \sum_{\alpha=1}^{2} N^{(\alpha)} U_1^{(\alpha)} + U_2^{(\alpha)} = 0 \\ \sum_{\alpha=1}^{2} L_{21} U_1^{(\alpha)} + L_{22} N^{(\alpha)} U_2^{(\alpha)} = 0 \end{aligned}\right\}. \tag{40}$$

If $(L_{12}+1)\ N^{(\alpha)}$ is assumed to be non-zero for $\alpha = 1, 2$, each ratio $U_2^{(\alpha)}/U_1^{(\alpha)}$ can be computed from the first of eqns (35): the system of eqn (40) may then be viewed as a linear system for the pair $(U_1^{(1)}, U_1^{(2)})$. The indeterminacy of this system yields:

$$[N^{(2)} - N^{(1)}][(\Delta - L_{22}X^2)N^{(1)}N^{(2)} - X^2(X^2 - L_{11})] = 0. \tag{41}$$

Alternatively, each ratio $U_1^{(\alpha)}/U_2^{(\alpha)}$ can be computed from the second of eqns (35) if $(1+L_{21})$ $N^{(\alpha)}$ is non-zero. Indeterminacy of the resulting linear system for the pair $(U_2^{(1)}, U_2^{(2)})$ yields:

$$[N^{(2)} - N^{(1)}][L_{22}X^2 N^{(1)}N^{(2)} - (X^2-1)(\Delta - L_{22}X^2)] = 0. \tag{42}$$

Leaving aside momentarily the possibility $N^{(2)} - N^{(1)} = 0$, eqns (41) and (42) may be combined to give:

$$N^{(1)}N^{(2)} = \frac{X^2(X^2 - L_{11})}{\Delta - L_{22}X^2} = \frac{(X^2-1)(\Delta - L_{22}X^2)}{L_{22}X^2} = [(1-L_{22})X^2 + (\Delta + L_{22} - L_{11})]\frac{X^2}{\Delta} - 1. \tag{43}$$

Thus the square of the speed of propagation-like scalar X solves the cubic equation:

$$\mathrm{F}(X^2) = c_0(X^2)^3 + c_1(X^2)^2 + c_2X^2 + c_3 = 0, \tag{44}$$

the coefficients c_0, c_1, c_2, and c_3 being given by

$$\left.\begin{aligned} c_0 &= L_{22}(1 - L_{22}) \\ c_1 &= L_{22}(L_{22} - L_{11} + 2\Delta) \\ c_2 &= -\Delta(2L_{22} + \Delta) \\ c_3 &= \Delta^2 \end{aligned}\right\}. \tag{45}$$

We consider now the special cases left aside so far: $N^{(1)} = N^{(2)} = N$ in eqn (41) or in eqn (42), and $(L_{12}+1)N^{(\alpha)} = 0$ or $(1+L_{21})N^{(\alpha)} = 0$ for some α in eqn (35). For that purpose, we mention that, throughout this study and for reasons explained in Section 3.3, the coefficients L_{11} and L_{22} will be required to be strictly positive.

In the first case, $N^{(1)} = N^{(2)} = N$ implies that there exists a single propagation mode that meets the boundary conditions of eqn (19). Then eqns (35) and (40) imply $X = 0$ together with $\Delta = 0$, $L_{12}+1 = 0$, $N^2 = L_{21}/L_{22} < 0$. Thus the single-mode kinematics may lead to a stationary surface wave $X = 0$ but under more restrictive conditions than the two-mode kinematics (see Section 3.4).

In the second case, $(L_{12}+1)N^{(\alpha)} = 0$ for one value of α is equivalent to $L_{12}+1 = 0$ for the mode α to be a surface mode; then eqn $(35)_1$ implies $(N^{(\alpha)})^2 = X^2 - L_{11}$ for both modes $\alpha = 1, 2$; thus either $N^{(1)} = N^{(2)}$, a possibility treated just above, or $N^{(1)} = -N^{(2)}$, but then one of the modes is not a surface mode. A similar conclusion holds for the possibility that $(1+L_{21})N^{(\alpha)} = 0$ for one value of α.

Now that we have established the equations for the main unknowns, namely, the propagation direction $\mathbf{n} = (n_1, n_2)$ and the wave-speed c, a few words are necessary to explain the analysis of instability developed in the sections that follow.

The general procedure and the main difficulty. Given the constitutive parameters, we first calculate the three squares of the wave-speeds by solving the cubic equation (44); for each solution, we calculate by eqn (36) the four associated directions of propagation, out of which we select, when they are available, those two that satisfy the decay condition, eqn (34); henceforth, these are denoted by $N^{(1)}$ and $N^{(2)}$. The key point is that these two solutions may not satisfy the boundary conditions at the free surface represented, for example, by the first equality of eqn (43). Indeed, it follows from the first two equalities in eqn (43) that $(N^{(1)}N^{(2)})^2$ is equal to P and so by construction the roots of eqns (36) and (37) always satisfy eqn (43) in absolute value. However, the signs of these two quantities implied by the decay condition of eqn (34) may be in contradiction with the signs involved in eqn (43). Thus, in order to check these equalities, it is sufficient to check the signs of $N^{(1)}N^{(2)}$ and, for example, the sign of the last term in eqn (43). This verification is greatly facilitated by the analytical tool developed in Appendix A.

The nature of the solutions. For solutions that satisfy the field and boundary equations, we shall delineate four situations depending on the nature of the square of the wave-speed c^2 or equivalently of X^2:

- if X^2 is real positive, the situation is similar to that of Rayleigh waves for an elastic solid, the solution being periodic in time (a *surface wave*);
- if X^2 is zero we have a *stationary surface wave*; notice that the onset of such a wave is *a priori* different from the onset of a stationary body wave since, for the latter, satisfaction of boundary conditions is not required;
- if X^2 is real strictly negative, the solution explodes as time elapses, and we have a *surface divergence instability*;
- if X^2 is complex, the solution grows while oscillating, a situation termed *surface flutter instability*.

These four situations correspond (cf. eqn (21)) to the existence of an eigenfunction $\mathbf{V} \in V$ ($\mathbf{V} \neq \mathbf{0}$) solution to Problem 2.1, such that $a(\mathbf{V}, \mathbf{V})$ is, respectively, *positive real*, *zero*, *negative real*, or *complex*.

The "control parameter" and the onset of dynamic surface instabilities. In commenting on our analysis, it will be assumed that the deformation processes begin when the material has elastic behaviour ($H = +\infty$ in eqn (29)) and that the modulus H (or equivalently the plastic modulus h) decreases continuously in the course of the deformation processes. The parameter H^{-1} thus plays the role of a "control parameter" that increases continuously from zero. Note also that the scalar Δ defined by eqn (38) has the value $\Delta = 4\ (\Lambda+1)$ at $H^{-1} = 0$ and decreases linearly with the "control parameter" H^{-1}. At the beginning of the deformation processes ($H^{-1} = 0$), the cubic equation (44) is nothing but the relevant cubic factor of the equation [cf. Fung (1965), eqn (13), p. 180, or eqn (C1) in Appendix C] that is obtained after squaring the characteristic Rayleigh equation [cf. Fung (1965), eqn (12), p. 180]. In that elastic situation, it is known that only one solution to eqn (44) (the unique real solution $X^2 = (c/c_S^e)^2 \in [0, 1]$) does solve the characteristic *Rayleigh equation* and indeed corresponds to a *surface wave* solution. As the "control parameter" H^{-1} is increased, we shall look for situations in which some X^2 solution of eqn (44) ceases to be a positive real number and the appropriate sign condition on $N^{(1)}N^{(2)}$ is satisfied: the onset of a *dynamic surface instability*. Dynamic surface instabilities are, of course, of interest when they precede possible body instabilities. We shall therefore restrict our analysis by *assuming that the equations of motion* (10) *are hyperbolic*: conditions for hyperbolicity of the field equations are derived in the next section.

Remark 3.1. By using the second equality in eqn (43), the product P (eqn (37)) can be recast in the following format:

$$P = \frac{1}{L_{22}}(X^2-1)(X^2-L_{11}) = \frac{(X^2)^2(X^2-L_{11})^2}{(\Delta - L_{22}X^2)^2}.$$

Hence, if the squares of the wave-speeds are real, i.e. X^2 is real according to (18), the values of X^2 are either smaller than min. $(1, L_{11})$ or larger than max. $(1, L_{11})$, since L_{22} will be required to be strictly positive as already alluded to; furthermore, P is positive or zero.

3.3. *Conditions for hyperbolic field equations*

We know that, for elastic–plastic solids with deviatoric associativity, loss of hyperbolicity coincides with the onset of a stationary body wave, that is c or $X = 0$, while at least one N^2 should be non-negative real, because, in an infinite body, only plane waves need to be considered. Thus, from eqn (36), hyperbolicity is retained if the inequalities below hold (SB is an acronym for a stationary body wave):

$$\left.\begin{array}{ll} \text{if } S_0 \geqslant 0\,, & \text{then } SB_0 \equiv P_0 - S_0^2/4 > 0\,; \\ \text{if } S_0 \leqslant 0\,, & \text{then } SB_0 \equiv P_0 > 0 \end{array}\right\}. \tag{46}$$

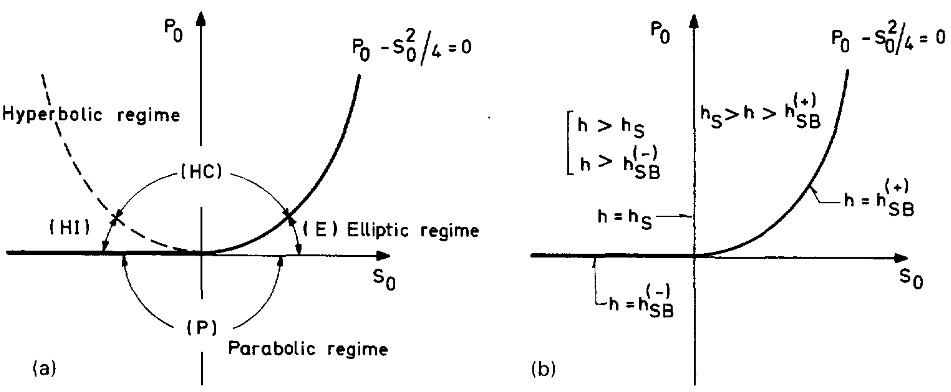
The onset of a stationary wave is then detected by the criterion (see Fig. 2a):

$$\left.\begin{array}{ll} \text{if } S_0 \geqslant 0\,, & \text{then } SB_0 \equiv P_0 - S_0^2/4 = 0\,; \\ \text{if } S_0 \leqslant 0\,, & \text{then } SB_0 \equiv P_0 = 0 \end{array}\right\}. \tag{47}$$

In eqns (46) and (47) and in the sequel, S_0 and P_0 denote the values of S and P in eqn (37) for $X = 0$. Note that, at the beginning of the deformation processes ($H^{-1} = 0$, $\Delta = 4(\Lambda+1)$), S_0 is equal to -2 while P_0 is equal to 1.

Let h_S be the value of the plastic modulus at which S_0 is zero. Using the constitutive equations, eqns (27), one obtains:

$$h_S = \frac{2\cos\psi\cos\chi}{\Lambda+2}\left[-(3\Lambda+4)\left[\hat{S}_{33} + \frac{N_\chi + N_\psi}{2(3\Lambda+4)}\right]^2 + (3\Lambda+4)\left[\frac{N_\chi+N_\psi}{2(3\Lambda+4)}\right]^2 + (\Lambda+1) - \frac{8}{3\Lambda+2}N_\chi N_\psi\right] \tag{48}$$



Sketch of the regimes of the field equations.

Fig. 2a. Definition of the hyperbolic (H), parabolic (P) and elliptic (E) regimes: (H): the two N^2 solutions of (36) are complex conjugate if $SB_0 = P_0 - S_0^2/4 > 0$ (HC) or real negative if $SB_0 = P_0 - S_0^2/4 \leqslant 0$ (HI); (P): one N^2 solution of (36) is real positive, the other one is real negative; (E): the two N^2 solutions of (36) are real positive. Fig. 2b. Characteristic moduli.

as well as the equivalence relation:

$$S_0 \lesseqgtr 0 \Leftrightarrow h \gtreqless h_S. \tag{49}$$

Now notice that, in the hyperbolic regime, $P_0 = L_{11}/L_{22}$ is restricted to strictly positive values. Since L_{11} and L_{22} take positive values for elastic behaviour and vary continuously with the plastic modulus, they must remain strictly positive in the hyperbolic regime:

$$L_{11} > 0, \quad L_{22} > 0. \tag{50}$$

Consequently it can be shown that, for $S_0 \geqslant 0$, $SB_0 > 0$ is equivalent to

$$(\hat{S}_{11} - \hat{S}_{22})^2 (h - h_{SB}^{(-)}) > 0,$$

where

$$h_{SB}^{(-)} = \frac{3\Lambda+2}{\Lambda+1} \cos\psi \cos\chi \left[-\left(\hat{S}_{33} + \frac{\tan\chi + \tan\psi}{2\sqrt{3}} \right)^2 + \frac{1}{3} \frac{(\Lambda+1)}{(\Lambda+2)} (\tan\chi - \tan\psi)^2 \right], \text{for } S_0 \geqslant 0. \tag{51}$$

For $S_0 \leqslant 0$, $SB_0 > 0$ is equivalent to

$$h - h_{SB}^{(-)} > 0,$$

where $h_{SB}^{(-)}$ is the value of the plastic modulus at which P_0 changes sign, i.e.

$$h_{SB}^{(-)} = \max_{k=1,2} \frac{4\cos\psi \cos\chi}{\Lambda+2} (N_\chi + \hat{S}_{kk})(N_\psi + \hat{S}_{kk}) - h_e \quad (\text{no sum on } k), \quad \text{for } S_0 \leqslant 0. \tag{52}$$

Notice that, for $\hat{S}_{11} = \hat{S}_{22}$, one has ($S_0 = -2, P_0 = 1$), as for elastic behaviour. Thus the conditions ensuring hyperbolicity, eqn (46), can be rephrased in terms of the plastic modulus h in the form:

$$h > h_{SB} \text{ with } h_{SB} \text{ defined by } \begin{Bmatrix} \text{eqn (51) if } h \leqslant h_S \\ \text{eqn (52) if } h \geqslant h_S \end{Bmatrix}, \tag{53}$$

while the onset of a stationary body wave is detected by $h = h_{SB}$ (see Fig. 2b).

In the next sections, we search for surface instabilities that may occur while the field equations are hyperbolic, that is, for a plastic modulus greater than the value h_{SB} defined in this section.

3.4. *Stationary surface waves and surface divergence instability*

3.4.1. *Stationary surface waves.* Stationary surface waves are available in the single-mode kinematics if the following conditions are satisfied (see the discussion of special cases in Section 3.2):

$$\Delta = 0, \quad L_{12} = -1, \quad N^2 = \frac{L_{21}}{L_{22}} < 0. \tag{54}$$

In the two-mode kinematics, insertion of the conditions $X^2 = 0$ in eqns (44) and (45) requires only $\Delta = 0$, the boundary conditions, eqns (41) and (42), are automatically satisfied and, from eqn (37):

$$P_0 - \frac{S_0^2}{4} = -\frac{1}{4L_{22}^2}(L_{12} - L_{21})^2 \leqslant 0.$$

Thus, in the hyperbolic regime, stationary surface waves can occur only in the HI regime (see Fig. 3) for non-associative plasticity, whereas, for associative plasticity, they can occur only along the parabola $P_0 - S_0^2/4 = 0, S_0 \leqslant 0$, the boundary of the (HI)–(HC) regimes.

Since Δ is linear with respect to H^{-1}, surface stationary waves will be observed in the HI regime if the plastic modulus h_{SS} such that $\Delta = 0$ (SS is an acronym for a stationary surface wave),

$$h_{SS} = \frac{3\Lambda+2}{\Lambda+1}\cos\psi\cos\chi\left[-\left(\hat{S}_{33} + \frac{\tan\chi+\tan\psi}{2\sqrt{3}}\right)^2 + \tfrac{1}{12}(\tan\chi-\tan\psi)^2\right], \tag{55}$$

is larger than the plastic modulus h_{SB} (eqn (52)) defining the onset of stationary body waves when $S_0 < 0$, and larger than the plastic modulus h_S defined by eqn (48) (see Fig. 2b). Thus, in the hyperbolic regime and since Δ is strictly positive for an elastic material,

$$\begin{cases} \Delta > 0 & \text{excludes stationary surface waves} \\ \Delta = 0 & \text{denotes the onset of stationary surface waves.} \end{cases} \tag{56}$$

Remark 3.2. Notice that, for associative plasticity, along the curve $\{SB_0 = P_0 - S_0^2/4 = 0, S_0 \geqslant 0\}$, the four solutions N to eqn (36) are real.

Remark 3.3. In both cases of associative and non-associative plasticity, it is not possible *a priori* to state if the onset of stationary surface waves always precedes or is always preceded by the onset of stationary body waves. In the *associative case* ($\psi = \chi$), we have (from eqns (51) and (55)) $h_{SS} = h_{SB}^{(+)}$ and (from eqns (52) and (55)):

$$h_{SS} - h_{SB}^{(-)} = \max_{k=1,2}\frac{\cos^2\chi}{(\Lambda+1)(\Lambda+2)}\{2N_\chi - [2(\Lambda+1)\hat{S}_{kk} + (\Lambda+2)\hat{S}_{33}]\}^2 \geqslant 0 \quad (\text{no sum on } k),$$

but it is not possible to order h_{SS} and h_S: that order depends on the values of $\psi = \chi$, Λ, and $\hat{\mathbf{S}}$. If $h_S \geqslant h_{SS} = h_{SB}^{(+)}$ then, when the monotonously decreasing plastic modulus h reaches the value $h_{SS} = h_{SB}^{(+)}$, we have $S_0 \geqslant 0$, so that (from Remark 3.2) the resulting stationary wave does not have a surface character. If $h_{SS} > h_S$ then, when the plastic modulus h reaches

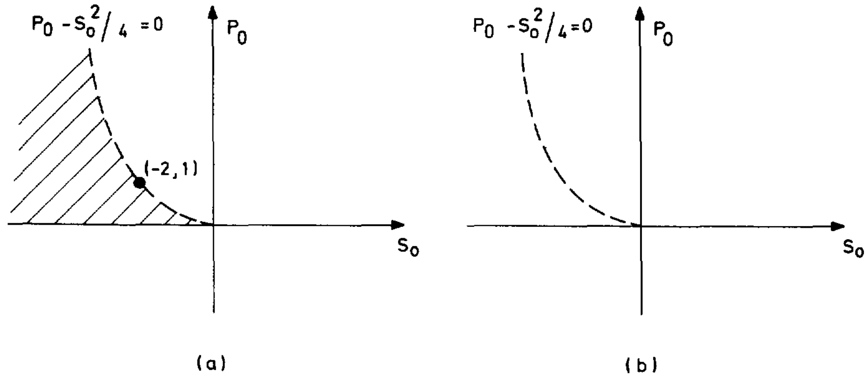


Fig. 3. Domains where surface stationary waves may be available in the hyperbolic regime: (a) for non-associative plasticity: $\{SB_0 = P_0 - S_0^2/4 < 0, P_0 > 0, S_0 \leqslant 0\}$; (b) for associative plasticity: $\{SB_0 = P_0 - S_0^2/4 = 0, S_0 \leqslant 0\}$.

the value $h_{SS}(>h_{SB}^{(-)})$, we have $S_0 < 0$ and the onset of stationary surface waves is reached before the onset of stationary body waves. In the *non-associative case*, we always have

$$h_{SB}^{(+)} - h_{SS} = \frac{(3\Lambda+2)^2}{12(\Lambda+1)(\Lambda+2)} \cos\psi \cos\chi (\tan\chi - \tan\psi)^2 \geqslant 0$$

but the order of $h_{SS}, h_{SB}^{(-)}$, and h_S depends on the values of the parameters ψ, χ, Λ, and $\hat{\mathbf{S}}$: as a consequence, the onset of stationary surface waves may precede or be preceded by the onset of stationary body waves.

Remark 3.4. *Further restriction of the investigation domain.* Henceforth we shall restrict the analysis by looking for surface instabilities that occur in the hyperbolic regime and before the onset of stationary surface waves, i.e. we consider plastic moduli larger than h_{SB} (eqn (53)) and h_{SS} (eqn (55).

3.4.2. *Surface divergence instabilities.* It may well happen that, during a deformation process, a solution satisfying the field and boundary conditions and associated with X^2 real and negative (surface divergence instability) is available in the hyperbolic regime. *We show, then, that this phenomenon cannot occur before the occurrence of stationary surface waves.*

The solutions to eqn (36) are given in Appendix A (cf. eqn (A5)): notice that both S and P are real and that P is positive according to Remark 3.1.

Proceeding as indicated in the Appendix, one can show that, for X^2 real negative, the product of the solutions $N^{(1)}$ and $N^{(2)}$ that satisfies the decay condition of eqn (34) is, in the hyperbolic regime, always equal to $-P^{1/2}$. On the other hand, the boundary conditions require satisfaction, in particular, of the second equation of eqns (43):

$$N^{(1)}N^{(2)} = \frac{(X^2-1)(\Delta - L_{22}X^2)}{L_{22}X^2}. \tag{57}$$

In the hyperbolic regime, L_{22} is positive (relations (50)), so that, for $\Delta > 0$ and $X^2 < 0$, the right-hand side of eqn (57) is positive, which contradicts the result $N^{(1)}N^{(2)} = -P^{1/2} < 0$.

Thus, as announced above, if available in the hyperbolic regime, stationary surface waves precede divergence instabilities. We therefore do not investigate the existence of the latter in more detail.

3.5. *Surface flutter instabilities*

We investigate the existence and onset of surface flutter instabilities under the restriction made in Remark 3.4. A surface flutter instability will be available if

- a root X^2 to eqn (44) is complex;
- the associated solutions $N^{(1)}$ and $N^{(2)}$ that satisfy the decay condition (eqn (34)) also satisfy the boundary condition (eqn (43)).

We now translate these requirements into an analytical form.

3.5.1 *Condition for equation* (44) *to admit complex roots.* On leaving aside momentarily the case $L_{22} = 1$ (see Remark 3.9) and taking into account eqn (50), the condition for the cubic equation (44) to admit complex roots is that its discriminant R be positive. It is shown in Appendix B that R can be cast in the form

$$R = \Delta^2 R_1(\Delta), \tag{58}$$

where $R_1(\Delta)$ is a polynomial in Δ. For $\Delta > 0$, the condition for eqn (44) to admit complex roots is thus $R_1 > 0$.

3.5.2. *Satisfaction of the boundary condition* (*equation* (43)). *A priori*, it may happen that, during a deformation process with decreasing plastic modulus, the boundary condition of eqn (43) is satisfied with R strictly positive. We shall prove that it is not so: the onset of flutter instability occurs on part of the domain $R = 0$. To prove this assertion, we first give a more tractable form to the sign problem to which satisfaction of the boundary condition, eqn (43), boils down (see discussion on Section 3.2). For that purpose, we shall make use of the results established in Appendix A.

We demonstrate the following.

Proposition 3.5. A necessary and sufficient condition for the onset of flutter instability is

$$R = 0, \quad BC > 0, \tag{59}$$

where

$$BC = 2\,\mathrm{Re}\,(X^2)(1-L_{22})+\Delta+L_{22}-L_{11}. \tag{60}$$

and Re (X^2) denotes the real part of the complex roots of the cubic equation (44), which, for $R = 0$, is a double root of that equation.

The proof of Proposition 3.5 is very technical. The scheme of the proof is as follows:

- first, we use the result established in Appendix C that shows that, in the elasticity case, BC is negative when the discriminant R is positive;
- we next show that, in the upper left quarter ($BC < 0, R > 0$) of the BC–R plane, the boundary condition cannot be satisfied (Lemma 3.6);
- thirdly, we show that, for $R = 0$, BC should be positive for the boundary condition to be satisfied (Lemma 3.7);
- finally, for $BC = 0$, we find that R cannot be positive (Lemma 3.8).

Thus the domain in which flutter may occur is contained in the upper right quarter ($BC > 0, R \geqslant 0$) and entrance in this domain is possible only along the boundary ($BC > 0, R = 0$) (see Fig. 4). To motivate this result, let us note that, through the last equality in eqn (43) and the definitions of S and P in eqns (37), we have:

$$\left.\begin{aligned} \mathrm{Im}(N^{(1)}N^{(2)}) &= \xi\frac{L_{22}}{\Delta}BC \\ \mathrm{Im}(S) &= \xi(1+L_{22}) \\ -\mathrm{Im}(P) &= \xi[1+L_{11}-2\,\mathrm{Re}\,(X^2)] \end{aligned}\right\}, \tag{61}$$

where $\xi \equiv \mathrm{Im}\,(X^2)/L_{22}$. We shall prove by using Appendix A that Im $(N^{(1)}N^{(2)})$ either has the sign of Im (S) or, if it has the sign of $-$Im (P), this quantity should be positive; both

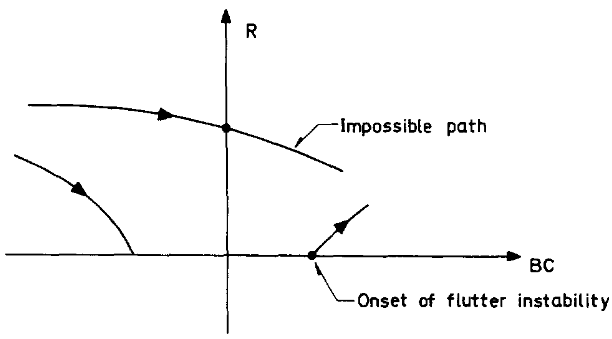


Fig. 4. Paths in the (BC, R)-plane during a deformation process.

cases result in $BC > 0$, since we are looking for the onset of flutter instability in the hyperbolic domain and before stationary surface waves, i.e. $L_{11} > 0$, $L_{22} > 0$ (relation (50)) and $\Delta > 0$ (Section 3.4).

We turn now to prove Lemmas 3.6–3.8 by making extensive use of the results established in Appendix A.

Lemma 3.6. For $BC < 0$ and $R > 0$, the boundary condition of eqn (43) cannot be satisfied.

It is shown in Appendix C that flutter is excluded for an elastic response because then the product $E^{(1)}E^{(2)}$ of the imaginary parts of the values of $N^{(\alpha)2}$ is positive, leading to Im $(N^{(1)}N^{(2)})$ and Im (S) being of the same sign (eqn (A7)); thus, from eqn (61), BC should be positive if eqn (43) were satisfied, whereas, in Appendix C, we actually find BC to be negative. Hence, during a deformation process, the single possibility to satisfy the boundary condition would be to have $E^{(1)}E^{(2)} < 0$, in which case Im $(N^{(1)}N^{(2)})$ and $-\text{Im}(P)$ are of the same sign (eqn (A12)) and the possibility $BC < 0$ may not be excluded. Because $E^{(1)}$ and $E^{(2)}$ are continuous functions of the modulus H, the above possibility necessarily entails passage through $E^{(1)} = 0$ *or* $E^{(2)} = 0$; the possibility $E^{(1)} = E^{(2)} = 0$ is excluded by eqn (A14) because then, according to eqn $(61)_2$, $\text{Im}(X^2)$ should be zero. But, if only one $E^{(\cdot)}$ is zero, then according to eqns (A15) and eqn $(61)_{1,2}$, BC should be positive, in contradiction to our assumption.

Lemma 3.7. For $R = 0$, BC should be positive.

To locate the double roots of the cubic polynomial F $(X^2) = 0$, we plot the first two of the three functions appearing in eqn (43), namely, with $Y = X^2$:

$$\mathrm{F}_1(Y) = \frac{Y(Y-L_{11})}{\Delta - L_{22}Y}, \quad \mathrm{F}_2(Y) = \frac{(Y-1)(\Delta - L_{22}Y)}{L_{22}Y},$$

$$\mathrm{F}_3(Y) = \frac{1-L_{22}}{\Delta}Y^2 + \frac{\Delta + L_{22} - L_{11}}{\Delta}Y - 1. \quad (62)$$

Although the proof below requires information on the signs of F_3 and $\mathrm{dF}_3/\mathrm{d}Y$ at double roots, there is no need to plot F_3. Indeed, if $\mathrm{F}_1 = N_1/D_1$ and $\mathrm{F}_2 = N_2/D_2$, then $\mathrm{F}_3 = (N_1+N_2)/(D_1+D_2)$; thus, at a point where $\mathrm{F}_1 = \mathrm{F}_2 = \mathrm{F}$ and $\mathrm{dF}_1/\mathrm{d}Y = \mathrm{dF}_2/\mathrm{d}Y = \Phi$, one has $\mathrm{F}_3 = \mathrm{F}$ and $\mathrm{dF}_3/\mathrm{d}Y = \Phi$. The plots in Fig. 5 delineate six cases according to the respective order of L_{11}, Δ/L_{22}, and 1. The limit cases where two or all of these quantities are equal do not introduce essentially new configurations. As expected, there is no intersection point in the interval] min.$(1, L_{11})$, max.$(1, L_{11})$[(see Remark 3.1). Notice that, in the associative case, Δ/L_{22} is always smaller than or equal to L_{11}, so that the three plots in Figs 5 (d), (e), and (f) corresponding to $L_{12}L_{21} < 0$ apply for non-associative plasticity only. Figure 5(a) is also representative of elastic behaviour.

The configurations Figs 5(a), (b), (e), and (f) may lead to the existence of a double root. We shall delineate the cases where the double root is smaller than $L_{11} < 1$ (Figs 5(e) and (f)) and where it is larger than max.$(1, L_{11})$ (Figs 5(a), (b), and (e)). In fact, we shall show that only the former situation may lead to the onset of flutter instability. Thus the value of the associated wave-speed c^{FLU} is always smaller than the elastic shear wave-speed $c_{\mathrm{S}}^{\mathrm{e}}$:

$$c^{\mathrm{FLU}} < \sqrt{L_{11}}\,c_{\mathrm{S}}^{\mathrm{e}} < c_{\mathrm{S}}^{\mathrm{e}}, \quad (63)$$

and we recover the result established in a more general context in Section 2.3 that flutter instability is excluded in the associative case.

Before we begin the proof of Lemma 3.7, it is interesting to note that there always exists a wave with wave-speed $c < c_{\mathrm{S}}^{\mathrm{e}}$ up to the onset of flutter, and it is precisely this wave that gives rise to the flutter phenomenon. Indeed, the existence of a root Y less than 1 is apparent from the plots in Fig. 5 and is not difficult to prove analytically through continuity

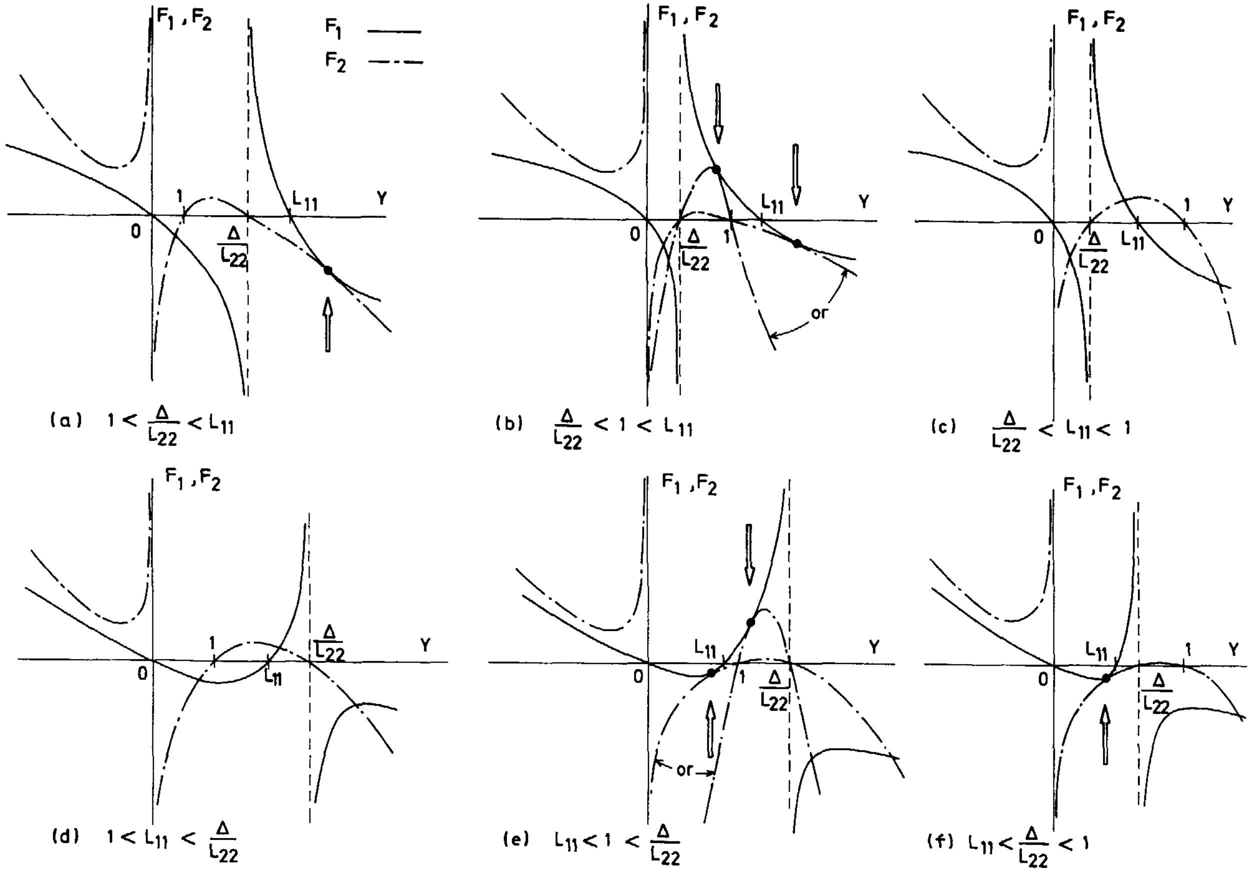


Fig. 5. Plots of functions $F_1(Y)$ and $F_2(Y)$ defining the boundary condition (43). Cases (d), (e), and (f) can occur only for non-associative plasticity. Case (a) is representative of elastic behaviour. Only the double root $Y < L_{11} < 1$, cases (e) and (f), is conducive to flutter instability.

arguments. However, this wave, which is a Rayleigh wave in the elastic case (see Appendix C), may cease to be a surface wave and may become a plane wave when $SB \equiv P - S^2/4 < 0$ and $S > 0$, a circumstance observed before the onset of flutter instability for particular material parameters. But, whether or not the surface wave character is lost, the boundary conditions of eqn (43) are always satisfied since the values of $N^{(1)}$ or $N^{(2)}$ are either both complex or both real: in the first case, $N^{(1)} N^{(2)}$ is negative according to eqns (A12)–(A14) as required, since $N^{(1)} N^{(2)}$ should be equal to F_3, which is negative after the plots in Fig. 5; in the second case, the signs of $N^{(\alpha)}$ are at our disposal and may be chosen to satisfy eqn (43). We now consider in turn the four possibilities delineated by Fig. 5.

(1) Let us consider first *the case of a double root*, $Y < L_{11}$ when $L_{11} < \min(\Delta/L_{22}, 1)$ (cf. Figs 5 (e) and (f)). According to eqn (61), Im (S) and $-$Im (P) are of the same sign, which, by using Appendix A, is also shared by Im $(N^{(1)} N^{(2)})$. According to the expression in eqn (61) for Im $(N^{(1)} N^{(2)})$, we deduce that, for the boundary condition of eqn (43) to be satisfied, BC in eqn (60) should be positive.

(2) *The case of a double root Y in the interval* $]\Delta/L_{22}, 1[$ (see Fig. 5(b)) does not lead to flutter. Indeed, we first notice that, at the double root, the slopes of the curves $\mathrm{F}_1(Y)$ and $\mathrm{F}_3(Y)$ are equal, leading to $\mathrm{dF}_3/\mathrm{d}Y < 0$ (see Fig. 5 (b)) and to $BC < 0$, because $BC = (\mathrm{dF}_3/\mathrm{d}Y)\,\Delta$. Now let us denote by $E^{(1)}$ and $E^{(2)}$ the imaginary parts of the solution N^2 to eqn (36). According to eqn (A7), if $E^{(1)}E^{(2)} > 0$, then satisfaction of the boundary condition of eqn (43) requires $BC > 0$ (cf. eqn $(61)_{1,2}$), which excludes flutter. If $E^{(1)}E^{(2)} < 0$, Re $(N^{(1)} N^{(2)}) = \mathrm{F}_3$ should be negative according to eqn (A13), excluding flutter for a double root in the interval $]\Delta/L_{22}, 1[$ (see Fig. 5(b)). Thus, whatever the sign of $E^{(1)}E^{(2)}$ is, flutter is excluded for a double root in $]\Delta/L_{22}, 1[$.

(3) We shall now prove that, *for a double root*, $Y > L_{11} > 1$ (see Figs 5 (a) and (b)), flutter is excluded because we have simultaneously $S > 0$, $SB < 0$, $BC < 0$ and $E^{(1)}E^{(2)} > 0$, a set of relations that is in contradiction with eqns (A7) and $(61)_{1,2}$. Using eqns (A2 and A4) with $SB < 0$, one has, just at the onset of flutter, i.e. for R slightly positive:

$$E^{(1)}E^{(2)} = \left(\frac{\mathrm{Im}\,S}{2}\right)^2 \frac{Z}{SB}, \quad Z(Y) \equiv \left(\frac{\mathrm{Im}\,P}{\mathrm{Im}\,S}\right)^2 - S\frac{\mathrm{Im}\,P}{\mathrm{Im}\,S} + P. \tag{64}$$

Thus it will be equivalent to prove that $S > 0$, $SB < 0$ and $Z < 0$, since $BC = (\mathrm{dF}_3/\mathrm{d}Y)\,\Delta$ is negative at $Y > L_{11}$ (see Figs 5(a) and (b)). Accounting for $L_{11} > 0$, $L_{22} > 0$, $\Delta > 0$, and eqns $(61)_{2,3}$, these inequalities are also equivalent to:

$$\left.\begin{aligned} \hat{S} &\equiv k + (1+L_{22})Y > 0 \\ \widehat{SB} &\equiv -(1-L_{22})^2 Y^2 - 2Y[2L_{22}(1+L_{11}) + k(1+L_{22})] + 4L_{11}L_{22} - k^2 < 0 \\ \hat{Z} &\equiv -(1-L_{22})^2 Y^2 - 2Y[2L_{22}(1+L_{11}) + k(1+L_{22})] + L_{22}(1+L_{11})^2 \\ &\quad + L_{11}(1+L_{22})^2 + k(1+L_{11})(1+L_{22}) < 0 \end{aligned}\right\} \tag{65}$$

with $k \equiv -\Delta + L_{12} + L_{21}$. Using the constitutive equations (27), one obtains:

$$\hat{S}(L_{11}) = (L_{11}-1)L_{22} + r(\Lambda+1)(\hat{S}_{11} - \hat{S}_{22})^2 > 0, \quad \text{for } L_{11} > 1. \tag{66}$$

Noticing that $\hat{Z}(L_{11})$ may be rewritten in the form

$$\hat{Z}(L_{11}) = (1-L_{11})[(1+L_{22})\hat{S}(L_{11}) + L_{22}(1-L_{11})] \tag{67}$$

and introducing the expression from eqn (65) of $\hat{S}(L_{11})$ into eqn (67), one readily deduces that $\hat{Z}(L_{11})$ is strictly negative for $L_{11} > 1$. It is also an easy matter to prove the inequalities

$$\left.\begin{aligned} &\widehat{SB}(L_{11}) = -(\hat{S}(L_{11}))^2 < 0 \\ &\frac{\mathrm{d}}{\mathrm{d}Y}\widehat{SB}(L_{11}) = \frac{\mathrm{d}}{\mathrm{d}Y}\hat{Z}(L_{11}) = -2[(1+L_{22})\hat{S}(L_{11})+2L_{22}L_{11}] < 0 \end{aligned}\right\}. \tag{68}$$

and this finishes the proof, since $\hat{S}$ is linear in Y, whereas $\widehat{SB}$ and $\hat{Z}$ are quadratic.

(4) Finally, we exclude flutter for *a double root* Y *in the interval* $]1, \Delta/L_{22}[$ *when* $L_{11} < 1$ (see Fig. 5(e)). Indeed, in this case, the slope of F_1, and thus BC, is positive at the double root. On the other hand, one can show that $\widehat{SB}$ (1) and d $\widehat{SB}$/dY (1) are negative here since $L_{11} < 1$:

$$\left.\begin{aligned} &\widehat{SB}(1) = -(\hat{S}(1))^2 < 0 \\ &\frac{\mathrm{d}}{\mathrm{d}Y}\widehat{SB}(1) = \frac{\mathrm{d}}{\mathrm{d}Y}\hat{Z}(1) = -2[(1+L_{22})\hat{S}(1)+2L_{22}] < 0 \end{aligned}\right\}. \tag{69}$$

Thus the quadratic function $SB(Y)$ is negative in the interval $]1, \Delta/L_{22}[$; hence, at the onset of flutter, $E^{(1)}\,E^{(2)}$ is negative because the roots $(N^{(\alpha)})^2$ are complex conjugate. But, if it is so, Re $(N^{(1)}\,N^{(2)})$ is negative according to eqn (A13); this is a contradiction with the fact that $N^{(1)}\,N^{(2)} = F_3$ and F_3 is positive in the interval of interest, as is clear in Fig. 5(e). As announced, flutter is thus excluded in this interval.

Thus, in summary, *the only case where the boundary condition is satisfied occurs when the double root* Y *is smaller than* 1 *while* $L_{11} < \min(\Delta/L_{22}, 1)$, an inequality that cannot occur for associative plasticity, *and then* BC *is positive.* □

Lemma 3.8. For $BC = 0$, the discriminant R is negative.

The proof is as follows: we assume first that R is positive and then we show that this cannot be true.

If R is positive, BC is well defined and may be rewritten in terms of constitutive coefficients only. By using eqn (B8), where the roots of the cubic are expressed in terms of the new coefficients u and v defined by eqn (B7), BC may be rewritten as:

$$BC = -\left[\frac{2}{3}\frac{c_1}{c_0}+u+v\right](1-L_{22})+\Delta+L_{22}-L_{11}. \tag{70}$$

The condition $BC = 0$ implies that the third root X_3^2 of the cubic equation (cf. eqn (B8)) is equal to

$$X_3^2 = -\frac{1}{3}\frac{c_1}{c_0}+u+v = -\frac{\Delta}{1-L_{22}}, \tag{71}$$

where the definitions (eqn (45)) of c_0 and c_1 have been used. Inserting this root into the cubic equation (44), we obtain:

$$\Delta = L_{11}L_{22}-1, \quad \text{or } L_{12}L_{21}-1 = 0. \tag{72}$$

The sum and product of the two other roots X_1^2 and X_2^2 are, respectively:

$$\left.\begin{aligned} X_1^2+X_2^2 &= -\frac{c_1}{c_0}-X_3^2 = -\left(\frac{\Delta+L_{22}-L_{11}}{1-L_{22}}\right) \\ X_1^2X_2^2 &= -\frac{c_1}{c_0}\frac{1}{X_3^2} = \frac{\Delta}{L_{22}} \end{aligned}\right\}, \tag{73}$$

where, once again, use of the definitions (eqn (45)) of c_0, c_1, and c_3 has been made. Accounting for eqns (72), one deduces that X_1^2 and X_2^2 are roots of the quadratic equation

$$(X^2)^2-(1+L_{11})X^2+\frac{L_{11}L_{22}-1}{L_{22}} = 0 \tag{74}$$

whose discriminant is found to be positive; thus the three roots X_1^2, X_2^2, and X_3^2 are real and the discriminant R of the cubic equation (44) should be negative, in contradiction with the assumption $R > 0$. □

Remark 3.9. The special case $L_{22} = 1$ does not introduce difficulties. Indeed, Proposition 3.5 still holds provided that R is set equal to the discriminant of the quadratic equation (44) with sign changed, i.e., $R = -\Delta^2\ (\Delta^2+4L_{12}\ L_{21})$. Moreover, double roots can occur only for non-associative flow rules since $L_{12}\ L_{21}$ should be negative.

3.6. *Quantitative discussion*

Since we have not been able to give an explicit expression for the plastic modulus at the onset of flutter instability, we perform a parameter analysis to detect directly the onset of flutter and to check the validity of Proposition 3.5. In this study, we vary, within physically acceptable limits and in a discrete but rather exhaustive manner, the Lamé modulus Λ, the triaxiality angle θ, the relative order between $\hat{S}_{11}$, $\hat{S}_{22}$, and $\hat{S}_{33}$, the angles χ and ψ, and the modulus H. The circumstances in which surface flutter instability occurs before the onset of other surface or body instabilities are analysed. Only the essential features of the results of that analysis are reported next.

It is observed that, in all the cases where the onset of flutter instability is reached, we always have $\hat{S}_{11} > \hat{S}_{22}$ (no restriction on the relative position of $\hat{S}_{33}$ is observed). We also observe that flutter instability is obtained for unusual (high) and very distinct friction and dilatancy angles. For instance, with the constitutive parameters:

$\Lambda = 0.25$
$\theta = 0.14(\pi/3)\quad (\hat{S}_{11} = 0.8078;\ \hat{S}_{22} = -0.3010;\ \hat{S}_{33} = -0.5068)$
and
$\tan\chi = 0.2$,

no flutter instability is detected before the onset of a stationary surface wave for any tan $\psi \in [0.2, 1.0]$ and $H^{-1} > 0$. Performing similar calculations with $\tan\chi = 0.4$, surface flutter instability is observed prior to other instabilities only when $\tan\psi = 0.9$ and $H^{-1} = 0.3917$; with $\tan\chi = 0.5$, surface flutter instability is detected when $\tan\psi = 0.8$ and $H^{-1} = 0.4114$.

As a result of this systematic numerical parameter analysis, it seems that flutter instability is excluded for the usual constitutive parameters.

It is also observed that, whenever the onset of flutter instability is reached, the (real) speed of the "prolongation" in the plastic range of the elastic Rayleigh wave coalesces with the (real) intermediate root of the cubic equation (44). At this stage, both solutions have a surface character, even if, along some phase of the incrementation process between $H^{-1} = 0$ and the critical value of H^{-1}, one or both of them do not have such a character (recall the comments preceding the proof of Lemma 3.7 above). The evolution of the three roots of

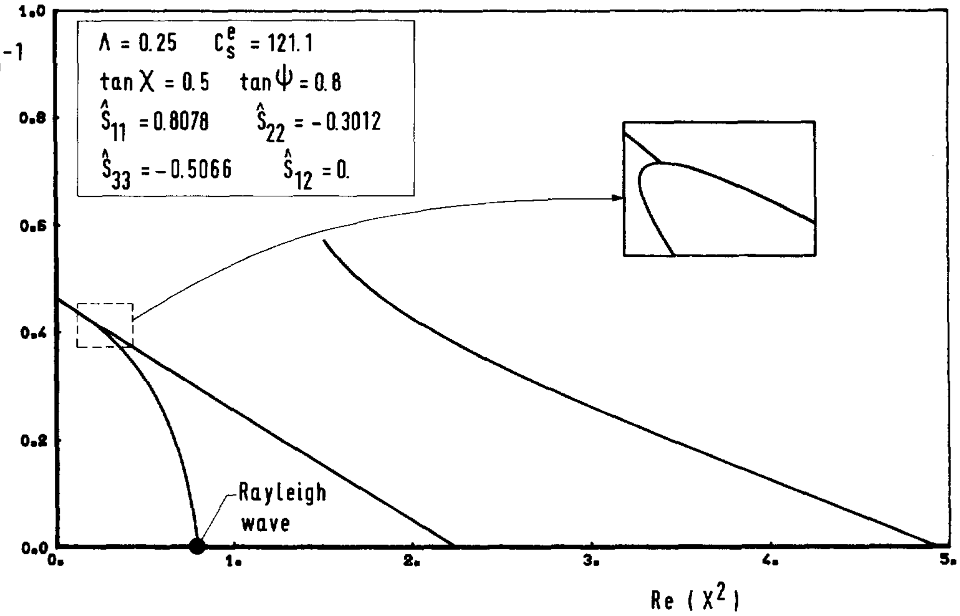


Fig. 6. Evolution of the three roots of the cubic equation (44) with H^{-1}.

the cubic equation (44) with H^{-1}, shown in Fig. 6 for the last case mentioned above (tan $\chi = 0.5$), is typical. At the onset of flutter instability, the two smaller roots of eqn (44) coalesce and, similarly to what happens in finite dimensional systems (Huseyin (1978)), the partial derivative of the "control parameter" H^{-1} with respect to X^2 is null at the onset of flutter. This property is implied by Proposition 3.5 and is due to the fact that $\hat{F}(X^2, H^{-1})$ defined by (eqn (44) is a polynomial with respect to both X^2 and H^{-1}. Thus, in the vicinity of a double root in X^2, $d\hat{F} = (\partial\hat{F}/\partial X^2)\, dX^2 + (\partial\hat{F}/\partial H^{-1})\, dH^{-1} = 0$ implies $\partial H^{-1}/\partial X^2 = 0$ because $\partial\hat{F}/\partial H^{-1}$ is not zero. The evolution in the complex plane of X^2, $N^{(1)}$, and $N^{(2)}$ for that case is shown in Figs 7 and 8 (case (a) $\hat{S}_{12} = 0$).

4. ELASTIC–PLASTIC SOLIDS NON-COAXIAL WITH THE FREE BOUNDARY

Non-coaxiality of the orthotropy axes and of the rate-traction-free boundary will be shown to give rise to flutter instability for non-associative flow rules right at the incipience of plasticity, i.e. for $H^{-1} \approx 0$. The variational formulation of Section 2.3 will be used to establish that the derivative dX^2/dH^{-1} at $H^{-1} = 0$ is complex if the normal to the boundary is not an axis of orthotropy of the constitutive tensor of plastic moduli (Section 4.2). A linearized analysis of the field equations will also give additional information on the mode shape, i.e. $dN^{(1)}/dH^{-1}$ and $dN^{(2)}/dH^{-1}$ at $H^{-1} = 0$ (Section 4.3). The evolution, with H^{-1} increasing monotonously from 0, of the triplet $(X^2, N^{(1)}, N^{(2)})$ is analysed quantitatively in Section 4.3 : it displays the respective influences of the features that trigger flutter instability, namely, deviation with respect to associativity and non-coaxiality with the boundary.

We also show in Section 4.5 that the onset of stationary surface waves is intrinsic, that is, it does involve material properties only and it does not discriminate among orientations of the rate-traction-free surface.

Notice that the theoretical results of Sections 4.2 and 4.5 do not require the constitutive equations to satisfy deviatoric associativity.

We first rewrite the constitutive equations in a format adequate to our present purpose.

4.1 *Constitutive equations*

Assuming as in Section 3 that the horizontal direction $\mathbf{e}_3$ is an orthotropy direction for the solid under consideration, the rate-constitutive equations (6) may be expressed in the $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$-axes in the form :

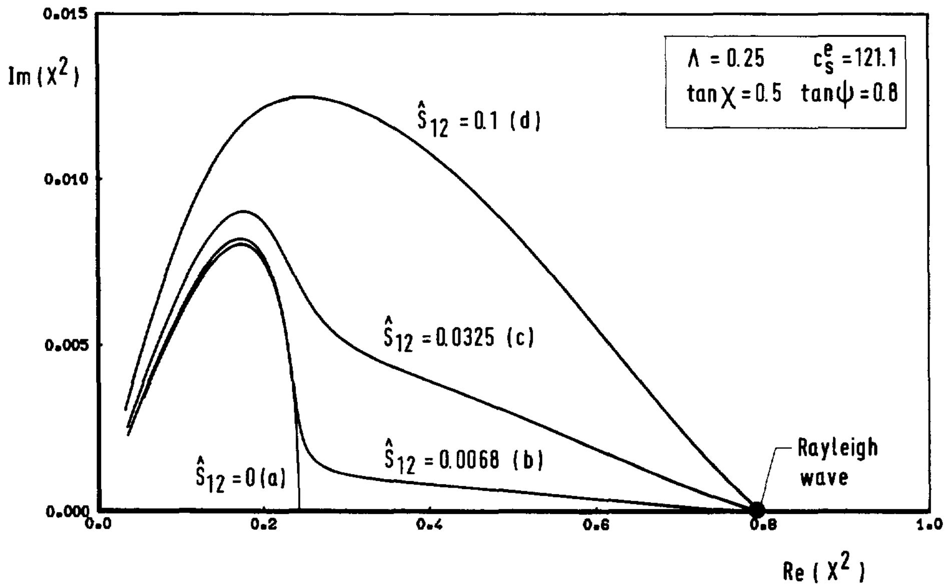


Fig. 7. Evolution in the complex plane of X^2 for several values of $\hat{S}_{12}$ and for H^{-1} varying from 0 to 0.458:
(a) $\hat{S}_{11} = 0.8078$, $\hat{S}_{22} = -0.3012$, $\hat{S}_{33} = -0.5066$, $\hat{S}_{12} = 0$.
(b) $\hat{S}_{11} = 0.807$, $\hat{S}_{22} = -0.301$, $\hat{S}_{33} = -0.506$, $\hat{S}_{12} = 0.0068$
(c) $\hat{S}_{11} = 0.81$, $\hat{S}_{22} = -0.30$, $\hat{S}_{33} = -0.51$, $\hat{S}_{12} = 0.0325$
(d) $\hat{S}_{11} = 0.8$, $\hat{S}_{22} = -0.3$, $\hat{S}_{33} = -0.5$, $\hat{S}_{12} = 0.1$.

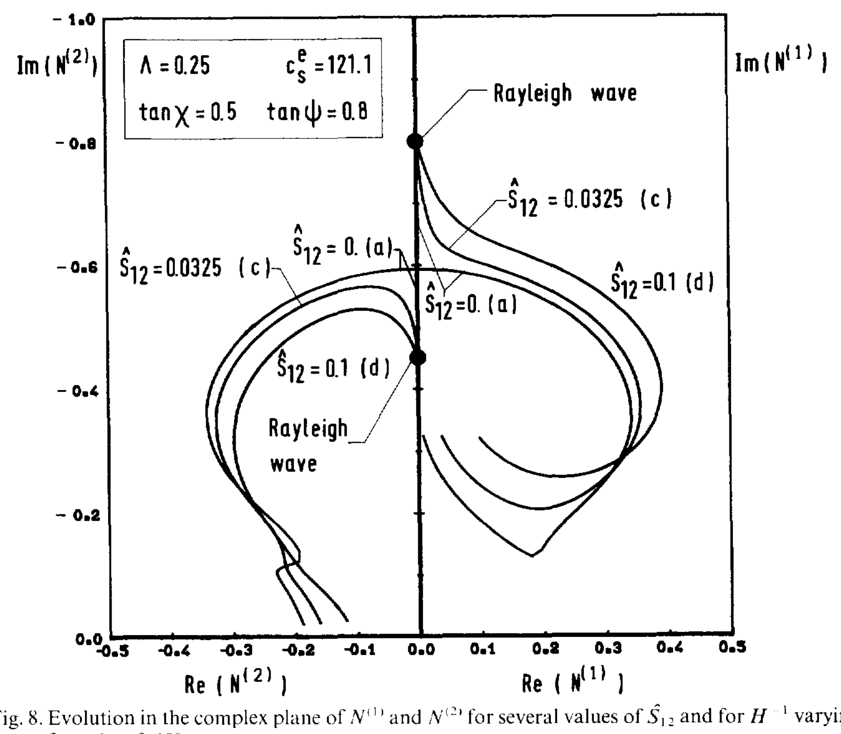


Fig. 8. Evolution in the complex plane of $N^{(1)}$ and $N^{(2)}$ for several values of $\hat{S}_{12}$ and for H^{-1} varying from 0 to 0.458:
(a) $\hat{S}_{11} = 0.8078$, $\hat{S}_{22} = -0.3012$, $\hat{S}_{33} = -0.5066$, $\hat{S}_{12} = 0$.
(c) $\hat{S}_{11} = 0.81$, $\hat{S}_{22} = -0.30$, $\hat{S}_{33} = -0.51$, $\hat{S}_{12} = 0.0325$
(d) $\hat{S}_{11} = 0.8$, $\hat{S}_{22} = -0.3$, $\hat{S}_{33} = -0.5$, $\hat{S}_{12} = 0.1$.

$$\left.\begin{aligned} \dot{\sigma}_{11} &= L_{11}\dot{\varepsilon}_{11} + L_{12}\dot{\varepsilon}_{22} + 2L_{13}\dot{\varepsilon}_{12} \\ \dot{\sigma}_{22} &= L_{21}\dot{\varepsilon}_{11} + L_{22}\dot{\varepsilon}_{22} + 2L_{23}\dot{\varepsilon}_{12} \\ \dot{\sigma}_{12} &= L_{31}\dot{\varepsilon}_{11} + L_{32}\dot{\varepsilon}_{22} + 2L_{33}\dot{\varepsilon}_{12} \end{aligned}\right\}. \tag{75}$$

Although the results obtained in this section do not require deviatoric associativity, one may identify the coefficients L_{ij} appearing above for an elastic–plastic solid obeying deviatoric associativity: the formulas in eqns (27)–(32) still hold and

$$\left.\begin{aligned} &\left.\begin{aligned} L_{i3} &= E_{ii12} \quad [= -r(N_\chi + \hat{S}_{ii})\hat{S}_{12}] \\ L_{3i} &= E_{12ii} \quad [= -r(N_\psi + \hat{S}_{ii})\hat{S}_{12}] \end{aligned}\right\} \quad i = 1, 2 \, (\text{no sum on } i) \\ &L_{33} = E_{1212} \quad [= 1 - r\hat{S}_{12}^2] \end{aligned}\right\}, \tag{76}$$

Trivially, these equations reduce to those of Section 3 if the horizontal axis $\mathbf{e}_1$ and the vertical axis $\mathbf{e}_2$ are orthotropy axes, i.e. if $\hat{S}_{12} = 0$.

4.2. *Surface flutter instability: a first general result*

The sesquilinear form of eqn (23) may be equivalently written in the form:

$$a(\mathbf{V}, \mathbf{W}) = \int_{-\infty}^{0} \overline{D_i W_j}\, E_{ijkm} D_m V_k \, \mathrm{d}x_2, \tag{77}$$

where

$$D_m = \mathrm{i} I_{1m} + I_{2m}(\,)'. \tag{78}$$

The constitutive tensor may be split into an elastic part, E^{e}_{ijkm}, which possesses the major symmetry $E^{\mathrm{e}}_{ijkm} = E^{\mathrm{e}}_{kmij}$, and a plastic part, E^{p}_{ijkm}, which in turn may be split into a symmetric part E^{sp}_{ijkm} and an anti-symmetric part E^{ap}_{ijkm} with respect to the pairs of indices ij and km:

$$E_{ijkm} = E^{\mathrm{e}}_{ijkm} + H^{-1} E^{\mathrm{p}}_{ijkm}, \tag{79}$$

$$E^{\mathrm{p}}_{ijkm} = E^{\mathrm{sp}}_{ijkm} + E^{\mathrm{ap}}_{ijkm}, \tag{80}$$

$$E^{\mathrm{sp}}_{ijkm} = (E^{\mathrm{p}}_{ijkm} + E^{\mathrm{p}}_{kmij})/2 = E^{\mathrm{sp}}_{kmij}, \tag{81}$$

$$E^{\mathrm{ap}}_{ijkm} = (E^{\mathrm{p}}_{ijkm} - E^{\mathrm{p}}_{kmij})/2 = -E^{\mathrm{ap}}_{kmij}. \tag{82}$$

Accordingly, the sesquilinear form of eqn (23) may be split into

$$a(\mathbf{V}, \mathbf{W}) = a^{\mathrm{e}}(\mathbf{V}, \mathbf{W}) + H^{-1} a^{\mathrm{sp}}(\mathbf{V}, \mathbf{W}) + \mathrm{i} H^{-1} a^{\mathrm{ap}}(\mathbf{V}, \mathbf{W}), \tag{83}$$

where

$$a^{\mathrm{e}}(\mathbf{V}, \mathbf{W}) = \int_{-\infty}^{0} \overline{D_i W_j}\, E^{\mathrm{e}}_{ijkm} D_m V_k \, \mathrm{d}x_2, \tag{84}$$

$$a^{\mathrm{sp}}(\mathbf{V}, \mathbf{W}) = \int_{-\infty}^{0} \overline{D_i W_j}\, E^{\mathrm{sp}}_{ijkm} D_m V_k \, \mathrm{d}x_2, \tag{85}$$

$$a^{\mathrm{ap}}(\mathbf{V},\mathbf{W}) = \int_{-\infty}^{0} \overline{D_i W_j} \frac{1}{\mathrm{i}} E^{\mathrm{ap}}_{ijkm} D_m V_k \,\mathrm{d}x_2. \tag{86}$$

Note that the sesquilinear forms $a^{\mathrm{e}}(\mathbf{V}, \mathbf{W})$, $a^{\mathrm{sp}}(\mathbf{V}, \mathbf{W})$, and $a^{\mathrm{ap}}(\mathbf{V}, \mathbf{W})$ are all symmetric in the sense of eqn (24); clearly, from the definitions of eqns (3), eqns (79)–(82), and eqns (85) and (86), both $a^{\mathrm{sp}}(\mathbf{V}, \mathbf{W})$ and $a^{\mathrm{ap}}(\mathbf{V}, \mathbf{W})$ are independent of H.

The eigenproblem (eqn (21)) may now be expressed as:

$$a^{\mathrm{e}}(\mathbf{V},\mathbf{W}) + H^{-1}a^{\mathrm{sp}}(\mathbf{V},\mathbf{W}) + \mathrm{i}H^{-1}a^{\mathrm{ap}}(\mathbf{V},\mathbf{W}) = X^2(\mathbf{V},\mathbf{W}), \forall \mathbf{W} \in V. \tag{87}$$

The elastic case is recovered by setting $H^{-1} = 0$ in eqn (87) and thus obtaining

$$a^{\mathrm{e}}(\mathbf{V}_{\mathrm{R}},\mathbf{W}) = X_{\mathrm{R}}^2(\mathbf{V}_{\mathrm{R}},\mathbf{W}), \forall \mathbf{W} \in V. \tag{88}$$

Taking into account eqns (22) and (84), the complex conjugate of eqn (88) yields

$$a^{\mathrm{e}}(\mathbf{W},\mathbf{V}_{\mathrm{R}}) = X_{\mathrm{R}}^2(\mathbf{W},\mathbf{V}_{\mathrm{R}}), \forall \mathbf{W} \in V. \tag{89}$$

In eqns (88) and (89), X_{R} and $\mathbf{V}_{\mathrm{R}}$ refer to the elastic Rayleigh wave-speed and eigenmode, respectively.

In order to study the eigenproblem (eqn (87)) in the neighbourhood of the (known) elastic case (eqn (88)), we compute the first variation of eqn (87) at $H^{-1} = 0$, all constitutive parameters being held fixed except H^{-1}:

$$\begin{aligned} &a^{\mathrm{e}}(\delta\mathbf{V},\mathbf{W}) + \delta(H^{-1})a^{\mathrm{sp}}(\mathbf{V}_{\mathrm{R}},\mathbf{W}) + \mathrm{i}\delta(H^{-1})a^{\mathrm{ap}}(\mathbf{V}_{\mathrm{R}},\mathbf{W}) \\ &\qquad = X_{\mathrm{R}}^2(\delta\mathbf{V},\mathbf{W}) + \delta(X^2)(\mathbf{V},\mathbf{W}), \quad \forall \mathbf{W} \in V. \end{aligned} \tag{90}$$

Setting $\mathbf{W} = \mathbf{V}_{\mathrm{R}}$ and using eqn (89), we obtain

$$\delta(H^{-1})[a^{\mathrm{sp}}(\mathbf{V}_{\mathrm{R}},\mathbf{V}_{\mathrm{R}}) + \mathrm{i}a^{\mathrm{ap}}(\mathbf{V}_{\mathrm{R}},\mathbf{V}_{\mathrm{R}})] = \delta(X^2)(\mathbf{V}_{\mathrm{R}},\mathbf{V}_{\mathrm{R}}), \tag{91}$$

and thus

$$\left(\frac{\mathrm{d}(X^2)}{\mathrm{d}(H^{-1})}\right)_{H^{-1}=0} = \frac{a^{\mathrm{sp}}(\mathbf{V}_{\mathrm{R}},\mathbf{V}_{\mathrm{R}})}{(\mathbf{V}_{\mathrm{R}},\mathbf{V}_{\mathrm{R}})} + \mathrm{i}\frac{a^{\mathrm{ap}}(\mathbf{V}_{\mathrm{R}},\mathbf{V}_{\mathrm{R}})}{(\mathbf{V}_{\mathrm{R}},\mathbf{V}_{\mathrm{R}})}. \tag{92}$$

For an orthotropic elastic–plastic solid with the direction $\mathbf{e}_3$ as a principal direction of orthotropy, the constitutive moduli involved in $a^{\mathrm{ap}}(\mathbf{V}_{\mathrm{R}}, \mathbf{V}_{\mathrm{R}})$ are given below (the corresponding values in the case of deviatoric associativity are also indicated in brackets):

$$E^{\mathrm{ap}}_{1122} = -E^{\mathrm{ap}}_{2211} = \frac{L_{12}-L_{21}}{2} \stackrel{\mathrm{def}}{=} \mathscr{F} \quad [= -4\cos\psi\cos\chi(N_\chi - N_\psi)(\hat{S}_{22}-\hat{S}_{11})]$$

$$\begin{aligned} &E^{\mathrm{ap}}_{1112} = -E^{\mathrm{ap}}_{1211} = \frac{L_{13}-L_{31}}{2} = E^{\mathrm{ap}}_{2212} = -E^{\mathrm{ap}}_{1222} = \frac{L_{23}-L_{32}}{2} \stackrel{\mathrm{def}}{=} \mathscr{E} \\ &\qquad\qquad \left[= -4\cos\psi\cos\chi(N_\chi - N_\psi)\frac{\hat{S}_{12}}{2}\right]. \end{aligned} \tag{93}$$

Note that all the equalities in eqn $(93)_2$ hold for any orthotropic elastic–plastic solid with the direction $\mathbf{e}_3$ as an orthotropy direction, even if deviatoric associativity does not hold. The sesquilinear form $a^{\mathrm{ap}}(\mathbf{V}_{\mathrm{R}}, \mathbf{V}_{\mathrm{R}})$ can then be expressed as

$$a^{\mathrm{ap}}(\mathbf{V}_{\mathrm{R}},\mathbf{V}_{\mathrm{R}}) = 2\mathscr{F}\int_{-\infty}^{0} \mathrm{Re}\left(\overline{D_1 V_{1\mathrm{R}}}\frac{1}{\mathrm{i}}D_2 V_{2\mathrm{R}}\right)\mathrm{d}x_2$$

$$+2\mathscr{E}\int_{-\infty}^{0} \mathrm{Re}\left[\overline{(D_1 V_{1\mathrm{R}}+D_2 V_{2\mathrm{R}})}\frac{1}{\mathrm{i}}(D_1 V_{2\mathrm{R}}+D_2 V_{1\mathrm{R}})\right]\mathrm{d}x_2. \quad (94)$$

Because $V_{1\mathrm{R}}$ is real and $V'_{2\mathrm{R}}$ is imaginary (see Appendix C), the coefficient of $\mathscr{F}$ in eqn (94) vanishes:

$$\mathrm{Re}\left(\overline{D_1 V_{1\mathrm{R}}}\frac{1}{\mathrm{i}}D_2 V_{2\mathrm{R}}\right) = -\mathrm{Re}\,(V_{1\mathrm{R}}V'_{2\mathrm{R}}) = 0. \quad (95)$$

The in-axis components of the tensor moduli therefore do not contribute to a^{ap}. In the coaxial case, $\mathscr{E}$ in (eqn (93)) is zero. Thus, in this case, we have $\mathrm{Im}\,\{\delta\,(X^2)\} = 0$, a result that complements the developments of Section 3.

Moreover, from eqns (C5) and (C6), we have:

$$\mathrm{Re}\left[\overline{(D_1 V_{1\mathrm{R}}+D_2 V_{2\mathrm{R}})}\frac{1}{\mathrm{i}}(D_1 V_{2\mathrm{R}}+D_2 V_{1\mathrm{R}})\right] = -(V'_{1\mathrm{R}}+\mathrm{i}V_{2\mathrm{R}})(V_{1\mathrm{R}}-\mathrm{i}V'_{2\mathrm{R}}). \quad (96)$$

Performing the integration of eqn (96) between $-\infty$ and 0 and taking into account the results in Appendix C, we obtain

$$a^{\mathrm{ap}}(\mathbf{V}_{\mathrm{R}},\mathbf{V}_{\mathrm{R}}) = 2\mathscr{E}\frac{U^2}{n_1}[1+(N_{\mathrm{R}}^{(1)})^2]\frac{N_{\mathrm{R}}^{(1)}-N_{\mathrm{R}}^{(2)}}{N_{\mathrm{R}}^{(1)}+N_{\mathrm{R}}^{(2)}}, \quad (97)$$

which is clearly different from zero in the non-coaxial case ($\mathscr{E}\neq 0$). Consequently, $\mathrm{Im}\,\{\delta(X^2)\}\neq 0$ in this case. This ends the proof of Proposition 4.1, which is stated below.

Proposition 4.1. For an orthotropic elastic–plastic half-space with one principal direction of orthotropy parallel to its boundary, the imaginary part of the derivative of the square of the wave-speed with respect to H^{-1} calculated at $H^{-1} = 0$ is different from zero if and only if the normal to the boundary of the half-space is not a principal direction of orthotropy and the flow rule is non-associative. □

4.3. *Field equations and linearization*

We seek solutions to the field equations that are obtained by superposition of surface waves of the form of eqn (33). Satisfaction of the field equations (14) results in a system of two linear equations for the components U_1 and U_2:

$$\left.\begin{aligned}[L_{11}+(L_{13}+L_{31})N+L_{33}N^2-X^2]U_1+[L_{13}+(L_{12}+L_{33})N+L_{32}N^2]U_2 &= 0\\ [L_{31}+(L_{21}+L_{33})N+L_{23}N^2]U_1+[L_{22}N^2+(L_{23}+L_{32})N+L_{33}-X^2]U_2 &= 0\end{aligned}\right\}. \quad (98)$$

Similarly to eqn (79), the dependence of the constitutive moduli on H^{-1} can be made explicit in eqn (98) by splitting the coefficients L_{ij} into elastic and plastic parts:

$$L_{ij} = L_{ij}^{\mathrm{e}}+H^{-1}L_{ij}^{\mathrm{p}}, \quad i,j = 1,2,3. \quad (99)$$

The vanishing of the determinant of eqn (98) yields a relation linking N, X^2, and the constitutive coefficients:

$$\phi(N, X^2, H^{-1}) = N^4 C_{11} + N^3[-C_{13} - C_{31}] + N^2[(C_{33} - C_{12} - C_{21}) - (L_{22} + L_{33})X^2]$$
$$+N[(C_{23} + C_{32}) - (L_{13} + L_{31} + L_{23} + L_{32})X^2] + C_{22} - (L_{11} + L_{33})X^2 + X^4 = 0, \quad (100)$$

where the coefficients C_{ij}, $i,j = 1$–3 are the cofactors of the constitutive matrix $\mathbf{L} = [L_{ij}]$.

Linearization of the above equation around each of the two Rayleigh solutions $N_R^{(\alpha)}$, $\alpha = 1, 2$, yields

$$\left(\frac{\mathrm{d}N^{(\alpha)}}{\mathrm{d}H^{-1}}\right)_{H^{-1}=0} = -\left(\frac{\dfrac{\partial\phi}{\partial H^{-1}} + \dfrac{\partial\phi}{\partial X^2}\dfrac{\partial X^2}{\partial H^{-1}}}{\dfrac{\partial\phi}{\partial N^{(\alpha)}}}\right)_{H^{-1}=0} \quad (101)$$

where $(\mathrm{d}X^2/\mathrm{d}H^{-1})_{H^{-1}=0}$ is given by eqn (92), and $(\partial\phi/\partial N^{(\alpha)})_{H^{-1}=0}$ can be checked to be different from zero for both $\alpha = 1$ and $\alpha = 2$. The detailed expressions for the quantities involved in eqn (101) are given in Appendix D.

Remark 4.2. Since the quartic equation (100) in N is not biquadratic, we are no longer sure that only two of its solutions satisfy the decay condition of eqn (34). However, in the elastic case ($H^{-1} = 0$) and for $X^2 = X_R^2$, it is known that a pair of solutions to eqn (100), $N_R^{(1)}$ and $N_R^{(2)}$, strictly satisfies and another pair strictly does not satisfy that decay condition. By continuity, the same situation then still holds for some range of small values of H^{-1}. Since, for this continuation of the Rayleigh solution, we already have flutter (Proposition 4.1), we shall not study what happens with the continuations of the other X^2 solutions to eqn (44) at $H^{-1} = 0$ that do not correspond to elastic surface solutions.

4.4. *Surface flutter instability: further quantitative results*

The boundary conditions of eqn (19) have been incorporated in the variational formulation. However, in order to perform a parameter analysis, we shall give them a tractable form in terms of the triplet (X^2, $N^{(1)}$, $N^{(2)}$). Indeed, proceeding as in Section 3.2, we obtain a linear system of two equations, which may be written, for instance, in terms of the components $U_1^{(\alpha)}$, $\alpha = 1, 2$:

$$\left.\begin{aligned}\sum_{\alpha=1}^{2}[L_{33}[N^{(\alpha)} + w^{(\alpha)}] + L_{32}N^{(\alpha)}w^{(\alpha)} + L_{31}]U_1^{(\alpha)} = 0\\ \sum_{\alpha=1}^{2}[L_{21} + L_{23}N^{(\alpha)}(L_{22}N^{(\alpha)} + L_{23})w^{(\alpha)}]U_1^{(\alpha)} = 0\end{aligned}\right\}, \quad (102)$$

with $w^{(\alpha)}$ obtained from the first of the field equations (98):

$$w^{(\alpha)} = \frac{U_2^{(\alpha)}}{U_1^{(\alpha)}} = -\frac{L_{11} + (L_{13} + L_{31})N^{(\alpha)} + L_{33}N^{(\alpha)2} - X^2}{L_{13} + (L_{12} + L_{33})N^{(\alpha)} + L_{32}N^{(\alpha)2}}. \quad (103)$$

Indeterminacy of the system of eqns (102) yields a relation linking X^2 and two symmetric functions of $N^{(1)}$ and $N^{(2)}$, $\Sigma = N^{(1)} + N^{(2)}$ and $\Pi = N^{(1)}N^{(2)}$. This relation is of the form

$$\gamma(N^{(1)}, N^{(2)}, X^2, H^{-1}) = c + bX^2 + a(X^2)^2 = 0, \quad (104)$$

where

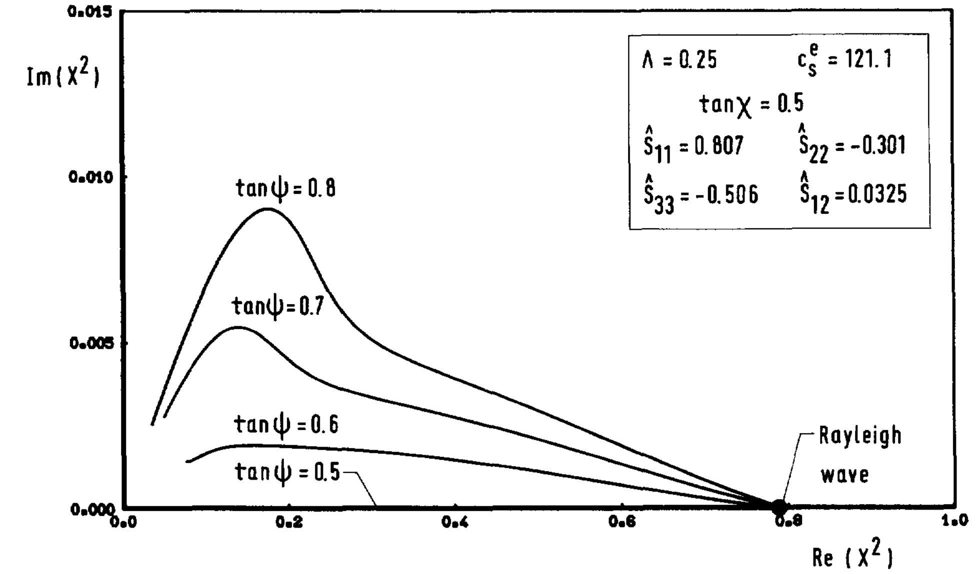


Fig. 9. Evolution in the complex plane of X^2 for different values of the non-associativity parameter $\tan\chi - \tan\psi$ and for H^{-1} varying from 0 to 0.458.

$$\left.\begin{aligned} a &= C_{11} \\ b &= -\Delta - L_{11}C_{11} + L_{33}C_{12} - L_{33}C_{11}\Sigma^2 + (L_{33}C_{13} - L_{13}C_{11})\Sigma + \\ &\quad [(L_{33} - L_{12})C_{11} + L_{32}C_{13}]\Pi - L_{32}C_{11}\Pi\Sigma \\ c &= -C_{11}C_{21}\Pi^2 + C_{11}C_{22}\Sigma^2 + (L_{13}\Delta - C_{13}C_{22})\Sigma + \\ &\quad [\Delta(L_{12} + L_{33}) - C_{13}C_{23} - 2C_{11}C_{22}]\Pi + C_{11}C_{23}\Pi\Sigma + L_{11}\Delta - C_{12}C_{22} \end{aligned}\right\}. \quad (105)$$

The numerical results presented in this section are obtained by incrementing H^{-1} (starting at the known elastic Rayleigh situation $(X_R^2, N_R^{(1)}, N_R^{(2)})$) and, at each stage, numerically solving the system of three non-linear algebraic equations:

$$\left.\begin{aligned} &\phi(N^{(1)}, X^2, H^{-1}) = 0 && \text{field equation for mode 1} \\ &\phi(N^{(2)}, X^2, H^{-1}) = 0 && \text{field equation for mode 2} \\ &\gamma(N^{(1)}, N^{(2)}, X^2, H^{-1}) = 0 && \text{boundary condition} \end{aligned}\right\} \quad (106)$$

for the triplet $(X^2, N^{(1)}, N^{(2)})$ with $\mathrm{Im}\,(N^{(\alpha)}) < 0$, for both $\alpha = 1$ and $\alpha = 2$. In Figs 7 and 8, we show the influence of the non-coaxiality parameter $\hat{S}_{12}$ on the evolution in the complex plane of X^2, $N^{(1)}$, and $N^{(2)}$. In contrast to the coaxial case ($\hat{S}_{12} = 0$), X^2 becomes complex right at the incipience of plasticity; the value of the imaginary part of X^2 near $H^{-1} = 0$ is roughly proportional to the parameter $\hat{S}_{12}$. In the non-coaxial case, the values of $N^{(1)}$ and $N^{(2)}$ acquire real parts at the incipience of plasticity, whereas, in the coaxial case, this happens later at a point that does *not* correspond to the onset of flutter; in the latter case, the onset of flutter occurs much later, at the point at which the equalities $\mathrm{Re}\,(N^{(1)}) = -\mathrm{Re}\,(N^{(2)})$ and $\mathrm{Im}\,(N^{(1)}) = \mathrm{Im}\,(N^{(2)})$ cease to hold. Figures 9 and 10 display the influence of the non-associativity parameter $\tan\chi - \tan\psi$ on the evolution in the complex plane of $(X^2, N^{(1)}, N^{(2)})$. Clearly, flutter instability does not occur in the associative case and, in the non-associative non-coaxial cases, the onset of flutter instability coincides with the incipience of plasticity. The value of the imaginary part of X^2 near $H^{-1} = 0$ is proportional to the difference $\tan\chi - \tan\psi$. Moreover, the values of $N^{(1)}$ and $N^{(2)}$ acquire a real part right at the incipience of plasticity even in the associative case, since, in this example, $\hat{S}_{12}$ is different from zero. A comparison between the linearized solution (eqns (92) and (101)) and the solution of the non-linear system of equations (106) is shown in

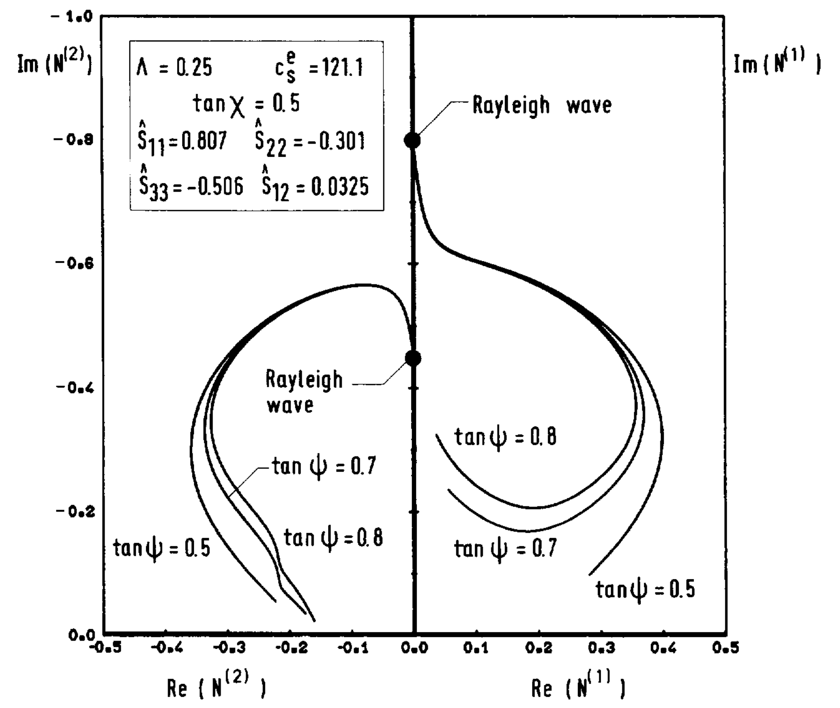


Fig. 10. Evolution in the complex plane of $N^{(1)}$ and $N^{(2)}$ for different values of the non-associativity parameter $\tan\chi - \tan\psi$ and for H^{-1} varying from 0 to 0.458.

Figs 11 and 12. Figures 7–10 display clearly the difference between the coaxial and the non-coaxial situations: in the coaxial case, flutter, if it occurs, appears well inside the plastic regime, whereas, in the non-coaxial case, it occurs right at the incipience of plasticity with intensity roughly proportional to the non-coaxiality and non-associativity parameters.

4.5 *Stationary surface waves*

In the present non-coaxial case, the field equations may be advantageously written in the orthotropy axes, say $(\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \mathbf{e}_3)$, β being the angle defined by $\tilde{\mathbf{e}}_1$ and $\mathbf{e}_1$ (see Fig. 13). Let

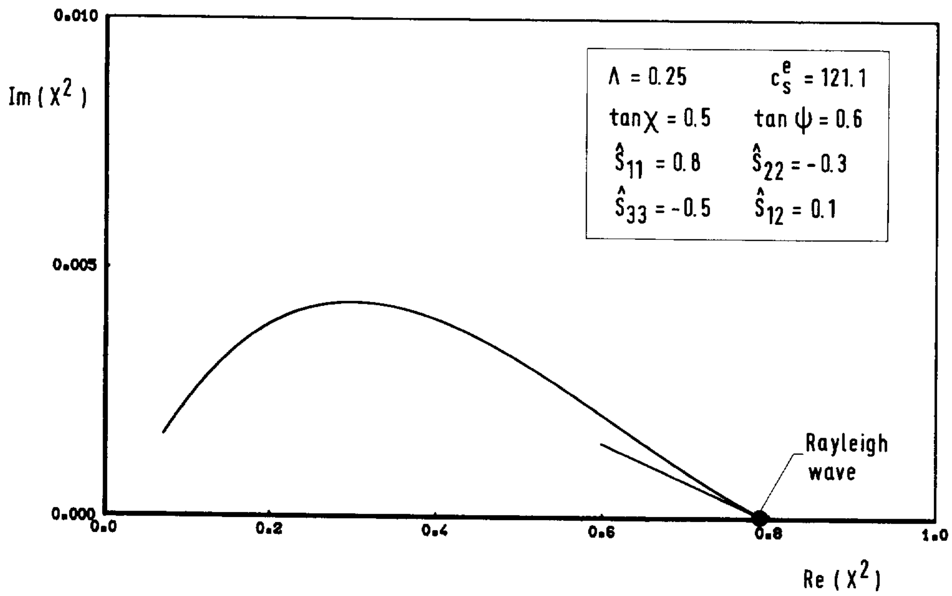


Fig. 11. Evolution in the complex plane of X^2. Comparison between the linearized solution and the solution of the non-linear equations.

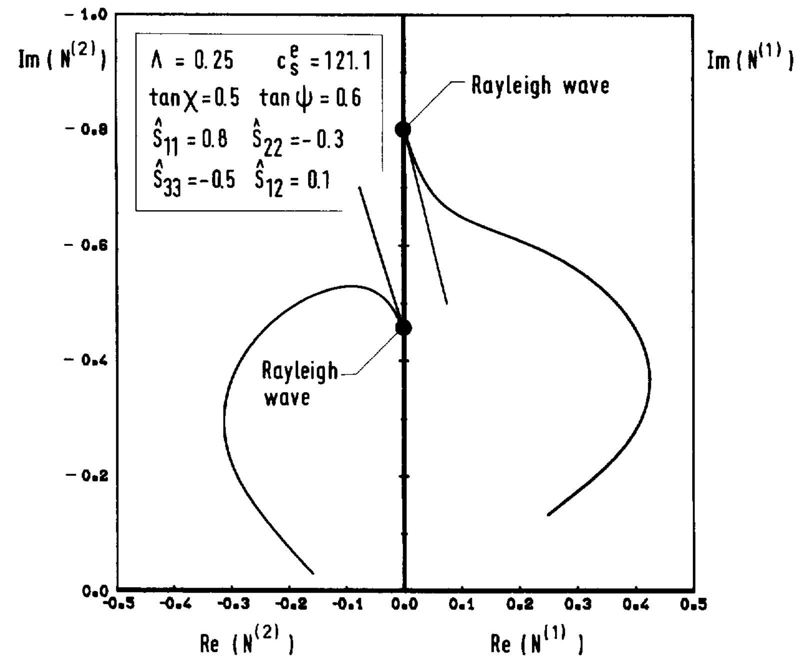


Fig. 12. Evolution in the complex plane of $N^{(1)}$ and $N^{(2)}$. Comparison between the linearized solution and the solution of the non-linear equations.

us recall that, for a solid with deviatoric associativity, the orthotropy axes are the axes of the deviator $\hat{\mathbf{S}}$ (eqn (32)). In these axes, the field equations are formally similar to the coaxial equations, that is

$$(\tilde{N}^2)^2 - \tilde{S}_0\tilde{N}^2 + \tilde{P}_0 = 0 \tag{107}$$

with $\tilde{N} = \tilde{n}_2/\tilde{n}_1$ and $\tilde{S}_0$ and $\tilde{P}_0$ given in terms of the constitutive coefficients $\tilde{L}_{ij}$ in the orthotropy axes:

$$\tilde{S}_0 = \frac{1}{\tilde{L}_{22}}(-\tilde{\Delta} + \tilde{L}_{12} + \tilde{L}_{21}), \quad \tilde{P}_0 = \frac{\tilde{L}_{11}}{\tilde{L}_{22}} \tag{108}$$

and

$$\tilde{\Delta} = \tilde{L}_{11}\tilde{L}_{22} - \tilde{L}_{12}\tilde{L}_{21}. \tag{109}$$

Since eqn (107) is biquadratic, two roots $\tilde{N}$ with negative imaginary parts and two roots with positive imaginary parts are available in the hyperbolic regime; if one assumes $\mathrm{Im}\,(\tilde{n}_1) = \mathrm{Im}\,(n_1) = 0$, then there exist two solutions $N = n_2/n_1$ with negative imaginary parts since $\mathrm{Im}\,(N) = \mathrm{Im}\,(\tilde{N})\tilde{n}_1/(\cos\beta \cdot n_1)$.

Now, observe that we can write the boundary conditions of eqn (9) in the orthotropy axes. These conditions may be recast in the single equation

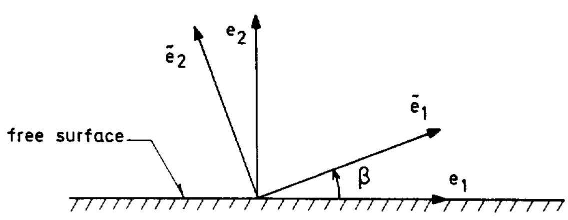


Fig. 13. Orthotropy axes.

$$\tilde{\Pi}(\cot^2\beta - \cot\beta\tilde{\Sigma} + \tilde{\Pi})\tilde{\Delta} = 0 \tag{110}$$

with $\tilde{\Pi} = \tilde{N}^{(1)}\tilde{N}^{(2)}$, $\tilde{\Sigma} = \tilde{N}^{(1)} + \tilde{N}^{(2)}$. Owing to the decay condition, $\tilde{\Pi}$ cannot be zero; similarly, in the hyperbolic regime, the second factor in eqn (110) can be shown to be non-zero. Thus the condition for the onset of a stationary surface wave is

$$\tilde{\Delta} = 0. \tag{111}$$

Moreover, even if the coefficient matrix $\mathbf{L}$ is not a tensor, $\Delta = \det \mathbf{L}$ is independent of β, i.e. $\tilde{\Delta} = \Delta$. Hence the onset of a stationary surface wave involves only material properties, that is, it does not discriminate among directions of free surfaces. This is in strong contrast to the situation that prevails at the onset of flutter instability as demonstrated above.

5. SUMMARY AND CONCLUSIONS

In the general context of semi-infinite elastic–plastic bodies with no traction rates applied on the boundary, it has been shown that, like body flutter instabilities, surface flutter instabilities are excluded for associative flow rules.

For elastic–plastic solids with deviatoric associativity, loss of hyperbolicity is known to coincide with the onset of stationary body waves, i.e. body flutter instability is excluded (Loret *et al.*, 1990). The following results were obtained when the directions defining the normal and the tangent to the rate-traction-free surface were axes of orthotropy of the constitutive tensor, or equivalently principal axes of the unit deviator $\hat{\mathbf{S}}$:

- the onset of stationary surface waves detected by $\Delta = 0$ (eqn (56)) may occur before or after the onset of stationary body waves detected by $SB_0 = 0$ (eqn (47));
- the onset of surface flutter instabilities that occur prior to both stationary body and surface waves is detected by the pair ($R = 0$, $BC > 0$) (eqn (59)), which implies in particular the existence of a double root to the cubic equation (44) for the square of the wave-speed-like quantity X^2 (eqn (18)); the associated wave-speed c is, like the Rayleigh wave-speed for elastic solids, smaller than the elastic shear wave speed c_S^e; indeed, loosely speaking, the surface flutter phenomenon is due to the instability of the prolongation well inside the plastic range of the elastic Rayleigh wave as explained in Section 3.5.2.

For elastic–plastic solids having orthotropic constitutive tensors and having one of the orthotropy axes tangent to the rate-traction-free boundary, it has been shown that

- whenever the normal to the rate-traction-free boundary does not coincide with one of the orthotropy directions of the constitutive tensor, the onset of surface flutter instabilities coincides with the incipience of plasticity ($H = +\infty$ in (eqn (3)) for any non-associative flow rule (cf. Proposition 4.1); in the deviatoric associativity case, this happens when the normal and the tangent to the boundary are not principal axes of the unit deviator $\hat{\mathbf{S}}$; in these circumstances, the Rayleigh wave becomes unstable as soon as the plastic range is entered; this result is in strong contrast with the result for the coaxial case above;
- the onset of stationary body waves involves material properties only, i.e. it does not discriminate among the orientations of the rate-traction-free boundary; this result is in contrast with the results concerning the onset of surface flutter instabilities, where the orientation of the boundary with respect to the material orthotropy axes is of utmost importance.

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APPENDIX A

Properties of the solutions of a biquadratic equation with complex coefficients

We are interested in the strictly complex solutions of the biquadratic equation with complex coefficients S and P:

$$(N^2)^2 - SN^2 + P = 0. \tag{A1}$$

Out of the four complex solutions to eqn (A1), we select those two, say, $N^{(1)}$ and $N^{(2)}$, whose imaginary parts are strictly negative. The sign of the imaginary part of the product $N^{(1)}N^{(2)}$, if it is not zero, will then be shown to be related to the sign of the imaginary part of either S or P.

In the proof below, $[.]^{1/2}$ denotes the square-root operator over positive real numbers. The analysis will make use of three coefficients A, B, and C:

$$\left.\begin{aligned} A &= \tfrac{1}{4}\,\mathrm{Re}\,(S)\,\mathrm{Im}\,(S) - \tfrac{1}{2}\,\mathrm{Im}\,(P)\,; \\ B &= \tfrac{1}{4}[\mathrm{Im}^2\,(S) - \mathrm{Re}^2\,(S)] + \mathrm{Re}\,(P)\,; \\ C &= \left[-\frac{B}{2} + \left\{\left(\frac{B}{2}\right)^2 + A^2\right\}^{1/2}\right]^{1/2} \end{aligned}\right\}. \tag{A2}$$

The squares of the solutions to eqn (A1) may be expressed by:

$$(N^{(\alpha)})^2 = D^{(\alpha)} + \mathrm{i}E^{(\alpha)}, \quad \alpha = 1, 2, \tag{A3}$$

where:

- if $A \neq 0$, $D^{(\alpha)} = \frac{1}{2}\mathrm{Re}(S) + \sigma C$; $E^{(\alpha)} = \frac{1}{2}\mathrm{Im}(S) + \sigma \frac{A}{C}$
- if $A = 0$ and $B \geqslant 0$, $D^{(\alpha)} = \frac{1}{2}\mathrm{Re}(S)$; $E^{(\alpha)} = \frac{1}{2}\mathrm{Im}(S) + \sigma[B]^{1/2}$
- if $A = 0$ and $B \leqslant 0$, $D^{(\alpha)} = \frac{1}{2}\mathrm{Re}(S) + \sigma[-B]^{1/2}$; $E^{(\alpha)} = \frac{1}{2}\mathrm{Im}(S)$. (A4)

where $\sigma = (-1)^{\alpha}$. Consequently, the four solutions to eqn (A1) can be cast in the form:

- if $E^{(\alpha)} \neq 0$, $N^{(\alpha)} = s^{(\alpha)}\left[F^{(\alpha)} + \mathrm{i}\dfrac{E^{(\alpha)}}{2F^{(\alpha)}}\right]$
- if $E^{(\alpha)} = 0$ and $D^{(\alpha)} \geqslant 0$, $N^{(\alpha)} = s^{(\alpha)}[D^{(\alpha)}]^{1/2}$
- if $E^{(\alpha)} = 0$ and $D^{(\alpha)} \leqslant 0$, $N^{(\alpha)} = \mathrm{i}s^{(\alpha)}[-D^{(\alpha)}]^{1/2}$, (A5)

where $s^{(\alpha)}$ can take the two values ± 1 and, for $E^{(\alpha)} \neq 0$, $F^{(\alpha)}$ is a strictly positive real number:

$$F^{(\alpha)} = [\tfrac{1}{2}[D^{(\alpha)} + G^{(\alpha)}]]^{1/2}, \quad G^{(\alpha)} = [(D^{(\alpha)})^2 + (E^{(\alpha)})^2]^{1/2}. \tag{A6}$$

Leaving apart real solutions, we select out of the four complex solutions those whose imaginary parts are strictly negative: notice that this selection criterion defines the scalars $s^{(\alpha)}$.

In order to prove the assertion announced at the beginning of this Appendix, we shall delineate four cases, as follows.

Case 1: $E^{(1)}E^{(2)} > 0$.

According to the selection criterion, we have $s^{(1)}s^{(2)} > 0$, and, from the expressions for $N^{(1)}$ and $N^{(2)}$ in eqn $(\mathrm{A5})_1$, the sign of $\mathrm{Im}\,(N^{(1)}N^{(2)})$ is equal to the sign of the $E^{(\alpha)}$'s, say, s. Scrutinizing the expressions for $E^{(\alpha)}$ in eqns (A4), we deduce that $\mathrm{Im}\,(S)$ is not zero and moreover $\mathrm{Im}\,(S) = s$. Thus we have:

$$E^{(1)}E^{(2)} > 0 \Rightarrow \mathrm{sign}\,\mathrm{Im}\,(N^{(1)}N^{(2)}) = \mathrm{sign}\,\mathrm{Im}\,(S). \tag{A7}$$

Case 2: $E^{(1)}E^{(2)} < 0$.

First let us recast $N^{(\alpha)}$ in the following format:

$$N^{(\alpha)} = \frac{s^{(\alpha)}G^{(\alpha)}}{2F^{(\alpha)}}[1 + \mathrm{e}^{\mathrm{i}\theta(\alpha)}], \tag{A8}$$

where the angle $\theta^{(\alpha)}$ is the argument of $(N^{(\alpha)})^2$; hence, from eqn (A3):

$$D^{(\alpha)} + \mathrm{i}E^{(\alpha)} = G^{(\alpha)}\,\mathrm{e}^{\mathrm{i}\theta(\alpha)}. \tag{A9}$$

Thus, since $s^{(1)}s^{(2)} = -1$, the imaginary part of $N^{(1)}N^{(2)}$ is equal to $-\sin\theta^{(1)} - \sin\theta^{(2)} - \sin\theta^{(1)}\cos\theta^{(2)} - \sin\theta^{(2)}\cos\theta^{(1)}$. Using the definitions of $\theta^{(\alpha)}$ (eqn (A9), of $D^{(\alpha)}$ and $E^{(\alpha)}$ (eqns (A4)), and of A (eqn $(\mathrm{A2})_1$), one can easily show that the sum of the last two terms is equal to:

$$u = \frac{-\mathrm{Im}\,(P)}{G^{(1)}G^{(2)}}. \tag{A10}$$

Furthermore the sum of the first two terms is equal to uv, where

$$v = \frac{E^{(1)}D^{(2)} - E^{(2)}D^{(1)}}{E^{(1)}G^{(2)} - E^{(2)}G^{(1)}}. \tag{A11}$$

Since $E^{(1)}$ and $E^{(2)}$ have opposite signs, the absolute value of v is strictly smaller than one. Consequently:

$$E^{(1)}E^{(2)} < 0 \Rightarrow \left\{\begin{matrix} \text{if } \mathrm{Im}\,(P) \neq 0, & \text{then sign Im}\,(N^{(1)}N^{(2)}) = -\,\text{sign Im}\,(P) \\ \text{if } \mathrm{Im}\,(P) = 0, & \text{then Im}\,(N^{(1)}N^{(2)}) = 0 \end{matrix}\right\}. \tag{A12}$$

Also notice that in this case:

$$\mathrm{Re}\,(N^{(1)}N^{(2)}) < 0. \tag{A13}$$

Case 3: $E^{(\alpha)} = 0$, $D^{(\alpha)} < 0$ for $\alpha = 1, 2$.

The analysis of the expressions of $E^{(\alpha)}$ and $D^{(\alpha)}$ (eqns (A4)) implies:

$$E^{(1)} = E^{(2)} = 0, D^{(1)} < 0, D^{(2)} < 0 \Rightarrow \mathrm{Im}\,(S) = 0 \text{ and } \quad N^{(1)}N^{(2)} = -[-D^{(1)}]^{1/2}[-D^{(2)}]^{1/2}. \tag{A14}$$

Case 4: $E^{(\alpha_1)} = 0$, $D^{(\alpha_1)} < 0$, $E^{(\alpha_2)} \neq 0$, $\alpha_1 \neq \alpha_2$.

Proceeding as in the previous case and scrutinizing the expression for $E^{(\alpha)}$ (eqns (A4)) one obtains $Im\,(S) = E^{(\alpha_2)}$, and analysis of the various possibilities yields the following:

$$E^{(\alpha_1)} = 0, \quad D^{(\alpha_1)} < 0, \quad E^{(\alpha_2)} \neq 0 \Rightarrow \mathrm{sign\,Im}\,(N^{(1)}N^{(2)}) = \mathrm{sign\,Im}\,(S) = \mathrm{sign}\,E^{(\alpha_2)}. \tag{A15}$$

APPENDIX B

Analysis of the cubic equations (44) *and* (45)

The cubic equation (44)

$$\mathrm{F}(Y) = c_0 Y^3 + c_1 Y^2 + c_2 Y + c_3 = 0 \tag{B1}$$

with coefficients defined by eqn (45), $c_0 \neq 0$, has two complex roots if the discriminant R is positive. The discriminant R may be defined either through two parameters p and q:

$$p = \frac{1}{3c_0^2}(3c_0c_2 - c_1^2), \quad q = \frac{1}{27c_0^3}(2c_1^3 - 9c_0c_1c_2 + 27c_0^2c_3) \tag{B2}$$

as

$$R = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2 \tag{B3}$$

or directly in terms of the coefficients c_k, $k = 0, 3$:

$$R = \frac{1}{108c_0^4}(-c_1^2c_2^2 + 4c_0c_2^3 + 27c_0^2c_3^2 + 4c_1^3c_3 - 18c_0c_1c_2c_3). \tag{B4}$$

Inserting the definitions of eqn (45) in this relation yields:

$$R = \frac{\Delta^2 L_{22}}{108c_0^4}[d_0\Delta^4 + d_1\Delta^3 + d_2\Delta^2 + d_3\Delta + d_4], \tag{B5}$$

where:

$$\left.\begin{aligned} d_0 &= -4 \\ d_1 &= 12L_{22} + 4L_{11}L_{22} \\ d_2 &= -12L_{22}^2 - 12L_{11}L_{22}^2 - 18L_{11}L_{22} - L_{11}^2L_{22} + 27L_{22} \\ d_3 &= 4L_{22}^3 + 12L_{11}L_{22}^3 + 20L_{11}^2L_{22}^2 - 36L_{11}L_{22}^2 \\ d_4 &= -4L_{11}L_{22}^4 + 8L_{11}^2L_{22}^3 - 4L_{11}^3L_{22}^2 < 0 \end{aligned}\right\}. \tag{B6}$$

When the discriminant R is positive, the roots Y_1, $Y_2 = \bar{Y}_1$ and Y_3 may be expressed through the two additional coefficients u and v satisfying $uv = -p/3$ and $u^3 + v^3 = -q$:

$$u = \left[-\frac{q}{2} + [R]^{1/2}\right]^{1/3}, \quad v = \left[-\frac{q}{2} - [R]^{1/2}\right]^{1/3} \tag{B7}$$

as follows:

$$\left.\begin{aligned} Y_1 &= -\frac{1}{3}\frac{c_1}{c_0} - \frac{u+v}{2} + \mathrm{i}\frac{[3]^{1/2}}{2}(u-v) \\ Y_2 &= -\frac{1}{3}\frac{c_1}{c_0} - \frac{u+v}{2} - \mathrm{i}\frac{[3]^{1/2}}{2}(u-v) \\ Y_3 &= -\frac{1}{3}\frac{c_1}{c_0} + u + v \end{aligned}\right\}. \tag{B8}$$

When the discriminant is zero while q is not, eqn (B1) admits a double root $Y_1 = Y_2$ and a simple root Y_3:

$$Y_1 = Y_2 = -\frac{1}{3}\frac{c_1}{c_0} - \frac{3}{2}\frac{q}{p}, \quad Y_3 = -\frac{1}{3}\frac{c_1}{c_0} + 3\frac{q}{p}. \tag{B9}$$

APPENDIX C

The (elastic) Rayleigh waves

The equations established in Section 3.2 and the tool developed in Appendix A may be applied to the purely elastic case to recover some results on the Rayleigh waves that are needed in several sections of the paper.

First, let us note that, since $L_{11} = L_{22} = \Lambda+2$, $L_{12} = L_{21} = \Lambda$, $\Delta = 4\,(\Lambda+1)$, and owing to the restriction (1) on the Lamé modulus Λ, one always has the situation depicted in Fig. 5(a), where $1 < \Delta/L_{22} \leqslant L_{11}$. The cubic equation satisfied by the wave-speed like scalar X^2 (eqn (44)), simplifies to:

$$\mathrm{F}(X^2) = (\Lambda+1)(\Lambda+2)\left[-(X^2)^3 + 8(X^2)^2 - 24\frac{\Lambda+4/3}{\Lambda+2}X^2 + 16\frac{\Lambda+1}{\Lambda+2}\right]. \tag{C1}$$

Its discriminant R, given by

$$R = \Delta^4[11(\Lambda+2)^3 - 62(\Lambda+2)^2 + 107(\Lambda+2) - 64], \tag{C2}$$

has a single root at $\Lambda_0 \cong 1.11$ and $R \lessgtr 0$ according to $\Lambda \lessgtr \Lambda_0$.

As is apparent from Fig. 5(a), there always exists a real positive root $X^2 < 1$; if $R < 0$, two additional real roots greater than L_{11} are available. Whatever the nature of X^2, one has, from eqns (37):

$$S = -2 + \frac{\Lambda+3}{\Lambda+2}X^2, \quad S^2 - 4P = \left(\frac{\Lambda+1}{\Lambda+2}X^2\right)^2 \tag{C3}$$

and, for a real X^2, P is positive (Remark 3.1). Hence for the real root $X^2 < 1$, S is negative, the values of $(N^{(\alpha)})^2$ are real negative, and the boundary condition of eqn (43) which requires $N^{(1)}N^{(2)} = \mathrm{F}_3\,(X^2) < 0$ (cf. Fig. 5 (a)), is satisfied. Thus the wave associated with this root is the surface wave with wave-speed $c < c_S^e$, usually termed the Rayleigh wave. The corresponding values of X^2 and $N^{(\alpha)}$ are identified by the subscript R and satisfy

$$N_R^{(1)} = -\mathrm{i}\sqrt{1 - \frac{X_R^2}{\Lambda+2}} \quad N_R^{(2)} = -\mathrm{i}\sqrt{1 - X_R^2}, \tag{C4}$$

while the corresponding eigenfunctions V_{kR} are

$$V_{1R}(n_1x_2) = U\left[\exp(\mathrm{i}N_R^{(1)}n_1x_2) + \left(\frac{X_R^2}{2} - 1\right)\exp(\mathrm{i}N_R^{(2)}n_1x_2)\right],$$

$$V_{2R}(n_1x_2) = UN_R^{(1)}\left[\exp(\mathrm{i}N_R^{(1)}n_1x_2) + \left(\frac{X_R^2}{2} - 1\right)^{-1}\exp(\mathrm{i}N_R^{(2)}n_1x_2)\right], \tag{C5}$$

where n_1 is a real number and, for our purposes, the arbitrary amplitude U may be chosen to be real.

For the two real roots X^2 greater than L_{11}, we have S, P, and $S^2 - 4P$ positive, so the values of $N^{(\alpha)2}$ are real positive and the associated wave is a plane wave.

Finally, let us consider the two conjugate complex roots X^2 when $R > 0$. Using the definitions of A, B, and C (eqn (A2)), one has:

$$A = \frac{k^2}{2}\mathrm{Im}\,\{(X^2)^2\}\,;\, B = -k^2\,\mathrm{Re}\,\{(X^2)^2\}\,;\, C = |k\,\mathrm{Re}\,(X^2)|\,;\, k = \frac{\Lambda+1}{2(\Lambda+2)}. \tag{C6}$$

We also have:

$$\frac{\mathrm{Im}\,(S)}{2} = \frac{\Lambda+3}{2(\Lambda+2)}\mathrm{Im}\,(X^2). \tag{C7}$$

Thus, according to eqn $(\mathrm{A4})_1$, the product $E^{(1)}E^{(2)}$ of the imaginary parts of the values (eqn (A3)) of $(N^{(\alpha)})^2$ is positive:

$$E^{(1)}E^{(2)} = \frac{\mathrm{Im}^2(X^2)}{\Lambda+2} > 0. \tag{C8}$$

Consequently, the boundary condition of eqn $(43)_3$ would require, according to eqns (A7), (C7), and (1), the ratio $BC = \{\mathrm{Im}\,(N^{(1)}N^{(2)})/\mathrm{Im}\,(X)^2\}\Delta$ to be positive:

$$BC = 2(1+\Lambda)[2-\mathrm{Re}\,(X^2)] > 0. \tag{C9}$$

Now, from eqn (C1), the sum of the roots $(X^2)_1+2\,\mathrm{Re}\,(X^2)_2$ with $(X^2)_1 < 1$ and $(X^2)_2 = \overline{(X^2)}_3$ complex is equal to 8, so that $\mathrm{Re}\,(X^2)_2 > 7/2$. Consequently, BC is negative and the boundary condition of eqn (43) is not satisfied, so that these two roots do not correspond to an actual surface wave.

APPENDIX D

Linearization of the field equation around the elastic solution

We differentiate eqn (100) at the single surface solution, which is available in the elastic situation (the Rayleigh wave). The implicit independent variable of the differentiation process is the inverse of the modulus H. The terms involved in eqn (101) are given below for deviatoric associativity:

$$\left(\frac{\partial f}{\partial H^{-1}}\right)_{H^{-1}=0} = 4\cos\chi\cos\psi$$

$$\begin{aligned}
&\{-N_{\mathrm{R}}^{(\alpha)^4}[L_{22}^{\mathrm{p}}+(\Lambda+2)L_{33}^{\mathrm{p}}]\\
&\quad+N_{\mathrm{R}}^{(\alpha)^3}[\Lambda(L_{23}^{\mathrm{p}}+L_{32}^{\mathrm{p}})-(\Lambda+2)(L_{13}^{\mathrm{p}}+L_{31}^{\mathrm{p}})]\\
&\quad+N_{\mathrm{R}}^{(\alpha)^2}[-(\Lambda+2)(L_{11}^{\mathrm{p}}+L_{22}^{\mathrm{p}})+(\Lambda+1)(L_{12}^{\mathrm{p}}+L_{21}^{\mathrm{p}})+2\Lambda L_{33}^{\mathrm{p}}+(L_{22}^{\mathrm{p}}+L_{33}^{\mathrm{p}})X_{\mathrm{R}}^2]\\
&\quad+N_{\mathrm{R}}^{(\alpha)}[-(\Lambda+2)(L_{23}^{\mathrm{p}}+L_{32}^{\mathrm{p}})+\Lambda(L_{13}^{\mathrm{p}}+L_{31}^{\mathrm{p}})+(L_{13}^{\mathrm{p}}+L_{31}^{\mathrm{p}}+L_{23}^{\mathrm{p}}+L_{32}^{\mathrm{p}})X_{\mathrm{R}}^2]\\
&\quad-[L_{11}^{\mathrm{p}}+(\Lambda+2)L_{33}^{\mathrm{p}}]+(L_{11}^{\mathrm{p}}+L_{33}^{\mathrm{p}})X_{\mathrm{R}}^2\},
\end{aligned} \tag{D1}$$

where the coefficients L_{ij}^{p} are defined by eqn (99),

$$\begin{aligned}
\left(\frac{\partial f}{\partial X^2}\right)_{H^{-1}=0} &= -(\Lambda+3)(1+N_{\mathrm{R}}^{(\alpha)^2})+2X_{\mathrm{R}}^2\\
&= \begin{cases} \dfrac{\Lambda+1}{\Lambda+2}X_{\mathrm{R}}^2 & \text{if}\,\alpha=1\\ -(\Lambda+1)X_{\mathrm{R}}^2 & \text{if}\,\alpha=2, \end{cases}
\end{aligned} \tag{D2}$$

$$\begin{aligned}
\left(\frac{\partial f}{\partial N^{(\alpha)}}\right)_{H^{-1}=0} &= 4(\Lambda+2)\,N_{\mathrm{R}}^{(\alpha)^3}+2[2(\Lambda+2)-(\Lambda+3)X_{\mathrm{R}}^2]N_{\mathrm{R}}^{(\alpha)}\\
&= (-1)^{\alpha}2(\Lambda+1)X_{\mathrm{R}}^2N_{\mathrm{R}}^{(\alpha)} \neq 0.
\end{aligned} \tag{D3}$$

To calculate

$$\left(\frac{\mathrm{d}X^2}{\mathrm{d}H^{-1}}\right)_{H^{-1}=0} = \frac{a^{\mathrm{sp}}(\mathbf{V}_{\mathrm{R}},\mathbf{V}_{\mathrm{R}})}{(\mathbf{V}_{\mathrm{R}},\mathbf{V}_{\mathrm{R}})}+\mathrm{i}\frac{a^{\mathrm{ap}}(\mathbf{V}_{\mathrm{R}},\mathbf{V}_{\mathrm{R}})}{(\mathbf{V}_{\mathrm{R}},\mathbf{V}_{\mathrm{R}})}, \tag{D4}$$

one needs:

$$(\mathbf{V}_{\mathrm{R}},\mathbf{V}_{\mathrm{R}}) = \frac{U^2}{\mathrm{i}n_1}\left(\frac{1}{2N_{\mathrm{R}}^{(1)}}-N_{\mathrm{R}}^{(1)}+\frac{N_{\mathrm{R}}^{(1)}}{2N_{\mathrm{R}}^{(2)^2}}+\frac{2N_{\mathrm{R}}^{(1)}}{\sqrt{-N_{\mathrm{R}}^{(1)}N_{\mathrm{R}}^{(2)}}}\right), \tag{D5}$$

$$\begin{aligned}
a^{\mathrm{sp}}(\mathbf{V}_{\mathrm{R}},\mathbf{V}_{\mathrm{R}}) = \frac{U^2}{\mathrm{i}n_1}&\left\{L_{11}^{\mathrm{p}}\left(\frac{1}{2N_{\mathrm{R}}^{(1)}}-\frac{N_{\mathrm{R}}^{(1)}}{2}-\frac{2\sqrt{-N_{\mathrm{R}}^{(1)}N_{\mathrm{R}}^{(2)}}}{N_{\mathrm{R}}^{(1)}+N_{\mathrm{R}}^{(2)}}\right)-L_{22}^{\mathrm{p}}\left(\frac{N_{\mathrm{R}}^{(1)^3}}{2}-\frac{N_{\mathrm{R}}^{(1)}}{2}+\frac{2N_{\mathrm{R}}^{(1)^2}\sqrt{-N_{\mathrm{R}}^{(1)}N_{\mathrm{R}}^{(2)}}}{N_{\mathrm{R}}^{(1)}+N_{\mathrm{R}}^{(2)}}\right)\right.\\
&+L_{33}^{\mathrm{p}}\left[-3N_{\mathrm{R}}^{(1)}+\frac{N_{\mathrm{R}}^{(1)}}{2N_{\mathrm{R}}^{(2)^2}}+\frac{N_{\mathrm{R}}^{(1)}N_{\mathrm{R}}^{(2)^2}}{2}+\frac{4\sqrt{-N_{\mathrm{R}}^{(1)}N_{\mathrm{R}}^{(2)}}}{N_{\mathrm{R}}^{(1)}+N_{\mathrm{R}}^{(2)}}\left(N_{\mathrm{R}}^{(1)}N_{\mathrm{R}}^{(2)}-\frac{N_{\mathrm{R}}^{(1)}}{N_{\mathrm{R}}^{(2)}}\right)\right]\\
&\left.+(L_{12}^{\mathrm{p}}+L_{21}^{\mathrm{p}})\left(N_{\mathrm{R}}^{(1)}+\frac{1-N_{\mathrm{R}}^{(1)2}}{N_{\mathrm{R}}^{(1)}+N_{\mathrm{R}}^{(2)}}\sqrt{-N_{\mathrm{R}}^{(1)}N_{\mathrm{R}}^{(2)}}\right)\right\}
\end{aligned} \tag{D6}$$

and $a^{\mathrm{ap}}(\mathbf{V}_{\mathrm{R}},\mathbf{V}_{\mathrm{R}})$, given by eqn (97).